Pricing & Hedging Options on Assets driven by Infinitely Divisible Vector Processes

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Abstract

In this paper, we introduce a class of quite general Lévy processes, with both a diffusion part and a pure jump component, as a prior distribution for log prices and volatilities in stochastic volatility models. This extends the work of Duffie *et al.* [2000] who model the jump part of the process as a compound Poisson process. Besides using a general Lévy process, we also allow dependence in jumps for both (log) prices and volatility. We show how to do option pricing using the change of measure required of us by The First Fundamental Theorem of Asset pricing (see Delbaen and Schachermayer [1994]), and we present other models in the literature as particular cases of our model. Finally, we outline a method for hedging European call options on assets driven by infinitely divisible vector processes.

Key words and phrases. Lévy processes; Option pricing; Risk-neutral measures; Mathematical Finance; Fast Fourier Transform; Stochastic Volatility.

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1. Introduction

Since the seminal papers of Black and Scholes [1973], Merton [1973], the option pricing literature has concerned itself with relaxing the key simplifying assumptions of that model, which include constant volatility, zero transactions costs and a flat yield curve. In our paper we use a framework which allows the relaxation of many of these classical assumptions at once, for example by incorporating stochastic volatility, a process for transactions costs, and a time-varying interest rate, in the determination of the option price. Unlike previous papers following a similar methodology (Duffie *et al.* [2000], Tavella and Randall [2000]), however, we derive the price and hedging parameters for a European call option when the underlying vector process above follows a Lévy process with a quite general specification. To this end, we begin our paper by introducing a class of Lévy processes with both a diffusion part and a pure jump component.

While jump-diffusion processes have been studied on several occasions before (see Duffie *et al.* [2000]), none of these processes are sufficiently general to address the issue of small jumps in the price of the underlying, the volatility, the interest rate, or transactions costs. However, these small jumps are potentially a large issue which should not be ignored. In fact, as demonstrated in Ait-Sahalia [2004], it is possible to distinguish statistically between (geometric) Brownian motion and jump processes even when there are infinitely many jumps. The use of Lévy processes exhibiting an infinite number of small jumps with discontinuous sample paths is not new and have been used in the seminal paper by Barndorff-Nielsen and Shephard [2001] as a way to model the stock price of an asset in a stochastic volatility context. Nicolato and Venardos [2003] considered the problem of pricing options on stock processes that follow the same probabilistical model as described in Barndorff-Nielsen and Shephard [2001]. In the same spirit, Cont and Tankov [2003] considered a wider range of such processes to price options by finding a risk-neutral measure either through the use of the Escher transform, relative entropy techniques or through the solution of a Partial Integro Differential Equation (PIDE) by invoking the Feynman-Kac theorem. This paper extends the work of Duffie *et al.* [2000] for pricing options through the resolution of a system of Riccatti equations, and can be seen as similar in spirit to the existing PIDE method from Cont and Tankov [2003].

The pure jump component, which we will refer to as a "pure Lévy" jump process, follows an infinitely divisible law. After presenting the conditional Fourier transform and its relation to option prices, we introduce the notation for our model in the case of two dimensions and show that two of the most important models in the literature, those of Duffie *et al.* [2000] and Barndorff-Nielsen and Shephard [2001], can be expressed as special cases. Following that, we present Theorem 1, which gives an explicit representation for the Ψ_t in the conditional Fourier transform. In practice, it is easier to perform the change to risk-neutrality in the space of characteristic functions than in the space of probability measures. However, because calculating option prices always requires explicit use of the probability measure for the underlying process, we must reconvert the characteristic function obtained using the above method back to the space of probability measures. Thus to price a European call option, we extend the Inversion Theorem of Duffie *et al.* [2000] to the case of a pure Lévy jump process. After describing how to compute the prices of the call option, we present a method for computing the "Greeks," ratios which traders use in practice to hedge the risk of portfolios containing the option in question. Our method allows us to combine our inverse Fourier transform result, which gives us the option price for given values of the

underlying vector X_t and values for the model parameters, with the finite difference method (see Tavella and Randall [2000]) over a grid of values for the parameters in the range of interest.
The outline of the paper is as follows. In section 1 we show how, by modelling the underlying as a general Lévy process, we can use the pricing methodology of Duffie *et al.* [2000] to compute option prices. In section 2, we show how the models of the underlying presented in Duffie *et al.* [2000], Barndorff-Nielsen and Shephard [2001] can be formulated as special cases of our infinitely divisible vector process framework. Section 3 derives the risk neutral parameters and presents a method for hedging European call options. Section 4 concludes.

1.1. The Basic Framework for Pricing European Call Options

In this paper we apply our technique to the pricing of a European call option. The holder of a *European call option* has the right, but not the obligation, to buy an underlying security at a specified date (*expiration date*) for a contractually specified amount (*strike price*), irrespective of the market value of the security on that date. [see Karatzas and Shreve 1997, p. 37]

The underlying securities of options can be stocks, indices such as the *Standard and Poor's 500*, interest rates, etc. It is standard in the financial literature to model stock prices on the log scale, i.e. set $S_t = \exp(X_t)$ and model X_t. At the expiration date T, the value of the option is $(S_T - K, 0)^+$, the maximum of $S_T - K$ and zero.
Payoff is at later time T, so under constant discount rate r present value of the call option at time t will be $\exp(-r(T-t))(S_T - K)^+$, where K is the strike price. If the discount rate varies with time t, perhaps as a function $R(X_t)$ of the log prices, the net present value is:

$$\exp\left(-\int_t^T R(X_s)ds\right)(S_T - K)^+$$

We are interested in the distribution of the discounted price S_T at time t, and can infer it from the conditional Fourier transform as in Duffie *et al.* [2000], which is defined as:

$$\Psi(u, X_t, t, T) \quad \equiv \quad E_t\{\exp\left(-\int_t^T R(X_s)ds\right)\exp\left(\langle u, X_T\rangle\right)\} \tag{1.1}$$

where E_t is the conditional expectation given the right-continuous $\mathbb{P}$-complete filtration $\mathcal{F}_t$, $\langle\rangle$ the inner product in $\mathbb{C}^n$ and $u \in \mathbb{C}^n$. For a didactic exposition of the conditional Fourier transform, please refer to the work of Duffie *et al.* [2000]. [see also Karatzas and Shreve 1997, p. 85 for a detailed mathematical explanation] .
If we want to solve (1.1) through the method proposed in this work, then $R(X_t)$ has to be an affine function[1]in X_t.
In order to price a derivative, using the risk-neutral approach might be too difficult due to the com-

[1]We say a function $f : \mathbb{R} \to \mathbb{R}$ is affine if there is a real-valued linear function $g(x)$ and a scalar $b \in \mathbb{R}$ such that $f(x) = g(x) + b$.

plexity of the processes X_t and so working with its conditional characteristic function (given by (1.1)) can be more tractable from a numerical and analytical point of view. Duffie *et al.* [2000] show that if $\int_{\mathbb{R}} |\Psi(a+ivb, x, 0, T)|dv < +\infty$, one can invert (1.1) in order to get its probability measure through the following equation:

$$G_{a,b}(y; X_t, T, Z) = \frac{\Psi(a, X_t, 0, T)}{2} - \frac{1}{\pi}\int_0^{+\infty} \frac{\Im[\Psi(a+ivb, X_t, 0, T)\exp(-ivy)]}{v} dv \tag{1.2}$$

where $\Im$ is the imaginary part of a complex number and $a, b \in \mathbb{R}$. Duffie *et al.* [2000] show that the price of a European call option is equal to:

$$C(a, c, T, Z) \quad = \quad G_{a,-a}(-\ln c; X_t, T, Z) - cG_{0,-a}(-\ln c; X_t, T, Z) \tag{1.3}$$

where $a \in \mathbb{R}$, c is the strike price, and X_t is given at time t. The computation of (1.2) is done through numerical methods if no analytical solution for it is available in closed form in order to compute (1.3).

1.2. Exposition of the Model

Black and Scholes [1973] modelled the stock price $S_t \in \mathbb{R}$ as Geometric Brownian motion that solves the following Stochastic Differential Equation (SDE):

$$\begin{aligned} dS_t &= S_t\mu dt + S_t\sigma dW_t \\ dS_t &= \mu(S_t)dt + \sigma(S_t)dW_t \end{aligned}$$

where $\mu(S_t) = S_t\mu$, $\sigma(S_t) = S_t\sigma$, and dW_t is Brownian motion. It is well known that Geometric Brownian motion does not induce in a satisfactory way stock returns to exhibit fat-tails, skewness and kurtosis, which are common features observed by the financial community. Different methods such as stochastic volatility together with Garch (which are both special cases of Brownian subordination[2], as well as the introduction of jumps in stock prices, are able to make the stock returns take the above features into account. We thus consider the case of a general vector-valued stochastic process $X_t \equiv [S_t, \sigma_t, \dots] \in \mathbb{R}^n$ that can integrate the stock process, stochastic volatility, interest rates as well as other univariate stochastic processes depending on the subjective taste and needs of the modeller in question, in order to price options on underlying S_t having other stochastic components that will also determine the risk-neutral measure (making S_t a martingale).

Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ be a probability space, equipped with a filtration satisfying the *usual hypotheses* [see Protter 1990, p. 3, for definition], and let X_t be a stochastic process with state space $S \subset \mathbb{R}^n$, Markov with respect to $(\mathcal{F}_t)_{t\geq 0}$,that solves the following SDE:

[2]Subordinating Brownian motion W_t with drift μ and volatility σ by a subordinator U_t yields a new Lévy process $Y_t = \sigma W(U_t) + \mu U_t$. [see Cont and Tankov 2003, ch. 4, for more on Brownian subordination]

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t + dJ_t \tag{1.4}$$

Here dW_t is an $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion in $\mathbb{R}^n$, and dJ_t is an $(\mathcal{F}_t)_{t\geq 0}$- pure jump Lévy process in $\mathbb{R}^n$ independent from dW_t, with intensity measure $\nu(dx)$ satisfying $\int_{\mathbb{R}^n}(\|x\|^2 \wedge 1)\nu(dx) < +\infty$.
The Lévy measure density $\nu(x)$ can be made affine in X_t, as in Kawazu and Watanabe [1971] in one dimension.
We use the parameterization[3] of Duffie *et al.* [2000], with drift and several other parameters of the model affine in X_t. Thus:

$$\begin{array}{rclllll}
\mu(x) & = & K_0 + K_1 x, & \text{where} & K & = & (K_0, K_1) \in \mathbb{R}^n \times \mathbb{R}^{n\times n} \\
(\sigma(x)\sigma^T(x))_{ij} & = & (H_0)_{ij} + (H_1)_{ij} x, & \text{where} & H & = & (H_0, H_1) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n\times n} \\
R(x) & = & \rho_0 + \rho_1 x, & \text{where} & \rho & = & (\rho_0, \rho_1) \in \mathbb{R} \times \mathbb{R}^n
\end{array}$$

The quantities [4] (K, H, ν, ρ) given X_0, characterize completely the distribution of X through the infinitesimal generator [5][Sato 1999, Ethier and Kurtz 1986]. We denote (K, H, ν, ρ) by Z below.
As in Duffie *et al.* [2000], (but less restrictive), define

A characteristic Z is *well-behaved* at $(u, T) \in \mathbb{C}^n \times [0, +\infty)$, if the following equations:

$$\dot{\beta} = \rho_1 - K_1^T \beta - \frac{1}{2}\beta^T H_1 \beta \tag{1.5}$$

$$\dot{\alpha} = \rho_0 - K_0^T \beta - \frac{1}{2}\beta^T H_0 \beta - \int_{\mathbb{R}^n} (\exp(\langle x, \beta\rangle) - 1 - \langle x, \beta\rangle I)\nu(dx) \tag{1.6}$$

are solved uniquely, and if $E(|\Psi_T|) < +\infty$, where $\Psi(u, X_T, T, T) \equiv \Psi_T$.
Very often, computing conditional Fourier transforms can not only be technically daunting, but impossible. This is why we derive the following theorem that allows us to find a closed-form solution of 1.1 in order to price options by using the Inversion Theorem of Duffie *et al.* [2000] stated previously.

Theorem 1 *If* Z *is well-behaved at* (u, T)*, then* $\alpha(t, T, u)$ *and* $\beta(t, T, u)$ *have a unique solution and we have the following equality,*

$$\Psi(u, X_t, t, T) \equiv E_t\{\exp\left(-\int_t^T R(X_s)ds\right)\exp(\langle u, X_T\rangle)\} = \exp\{\alpha(t, T, u) + \beta(t, T, u)X_t\} \tag{1.7}$$

[3]The parameters of the model are called the characteristics.
[4]We shall assume that H_0 is symmetric as in Duffie *et al.* [2000].
[5]The infinitesimal generator is a linear operator $\mathcal{D}$ such that $\left\{f(X_t) - \int_0^t \mathcal{D}f(X_s)ds\right\}_{t\geq 0}$ is a martingale for any function f in its domain, and is fully determined by the drift, volatility and other characteristics of X_t.

with boundary conditions $\alpha(T,T,u)=0$ *and* $\beta(T,T,u)=u$.

Theorem 1 thus gives an explicit way of computing the conditional Fourier transform $\Psi(u, X_t, t, T)$ given by 1.1 through a system of complex-valued Ricatti equations for $\alpha(t,T,u)$ and $\beta(t,T,u)$ given by 1.5 and 1.6. The proof of theorem 1 is given in the appendix.

2. Special Cases of the Model

In the previous section, we found how to compute the conditional Fourier transform through theorem 1 for a process $X_t \in \mathbb{R}^n$ satisfying equation 1.4. The change of probability measure to a risk-neutral one in order to price options will be developed in section 3. Before doing this, however, we investigate how Barndorff-Nielsen and Shephard [2001] and Duffie *et al.* [2000] are special cases of equation 1.4, which enables one to price options on underlyings following these specifications.

2.1. Barndorff-Nielsen and Shephard's model

Barndorff-Nielsen and Shephard [2001] consider a related model, which in our notation can be described as:

$$\begin{aligned}
X_t &= \begin{pmatrix} Y_t \\ \sigma_t^2 \end{pmatrix} \\
\mu(X_t) &= K_0 + K_1 X_t = \begin{pmatrix} \mu \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & \beta \\ 0 & -\lambda \end{pmatrix} \begin{pmatrix} \log S_t \\ \sigma_t^2 \end{pmatrix} \\
\sigma(X_t) &= \sqrt{\sigma_t^2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
J_t &\sim \text{Lévy}\big(\nu(dx)\big)
\end{aligned}$$

where J_t is a Lévy process with Lévy measure

$$\nu(dx) \;=\; \frac{\lambda\delta}{\sqrt{8\pi}}(x_2^{-1}+\gamma^2)x_2^{-\frac{1}{2}}\exp(-\frac{\gamma^2 x_2}{2})\delta_{\rho x_2}(dx_1)\,dx_2,$$

concentrated on the line $x_1 = \rho x_2$ so that each jump in asset log price is a constant multiple ($\rho < 0$) of a corresponding jump in volatility. Barndorff-Nielsen and Shephard [2001] denote J_t^2 by Z_t and J_t^1 by ρZ_t, and note that Z_t is a real-valued subordinator, i.e., an increasing Lévy process with no Gaussian component.

In their notation our Equation 1.4 becomes:

$$\begin{aligned} dY_t &= (\mu + \beta\sigma_t^2)dt + \sigma_t dW_t + \rho dZ_{\lambda t} \\ d\sigma_t^2 &= -\lambda\sigma_t^2 dt + dZ_{\lambda t} \end{aligned}$$

Here $\mu + \beta\sigma_t^2$ is the risk premium in time dt, decomposable as a fixed part μ, and another that based on the risk $\beta\sigma_t^2$ due to volatility.
This model has a *leverage effect* [see Barndorff-Nielsen and Shephard 2002, p. 19, for definition] in which a drop in the price will lead to higher volatility (since $\rho \leq 0$).
The time change of Z_t to $Z_{\lambda t}$ is just to simplify formulas by making the invariant distribution of σ_t^2 not depend on λ.
The volatility process σ_t^2 is of the O-U type, and the driving process Z_t is not Brownian motion, but a subordinator, and admits an invariant distribution (see Sato [1999], section 17).
The stochastic process Z_t is called a *background driving Lévy process* (BDLP). See Barndorff-Nielsen and Shephard [2001] for more details.

2.2. Duffie and Pan's model

The model of Duffie *et al.* [2000] can be expressed in our notation as:

$$\begin{aligned} X_t &= \begin{pmatrix} Y_t \\ V_t \end{pmatrix} \\ \mu(X_t) &= K_0 + K_1 X_t = \begin{pmatrix} r - \bar{\xi} - \bar{\lambda}\bar{\mu} \\ \kappa_v \bar{v} \end{pmatrix} + \begin{pmatrix} 0 & -\frac{1}{2} \\ 0 & -\kappa_v \end{pmatrix} \begin{pmatrix} \log S_t \\ V_t \end{pmatrix} \\ \sigma(X_t) &= \sqrt{V_t} \begin{pmatrix} 1 & 0 \\ \bar{\rho}\sigma_v & \sqrt{1 - \bar{\rho}^2 \sigma_v} \end{pmatrix} \\ J_t &\sim \text{Lévy}\big(\nu(dx)\big) \end{aligned}$$

where J_t (denoted Z_t by Duffie *et al.* [2000]) has bivariate Lévy density:

$$\nu(x_1, x_2) = \frac{\lambda}{\sqrt{2\pi}\sigma_{x_1}\mu_{x_2}} \exp\left(-\frac{x_2}{\mu_{x_2}} - \frac{(x_1 - \mu_{x_1} - \rho_J x_2)^2}{2\sigma_{x_1}^2} \right), \qquad x \in \mathbb{R} \times \mathbb{R}_+.$$

Here $x_1 \in \mathbb{R}$, the size of the jump in the log price, follows a normal distribution. It is correlated (through ρ_J) with the exponentially-distributed size $x_2 > 0$ of the jump in volatility. Finally λ is the intensity of the Poisson process. See Duffie *et al.* [2000] for details.
Duffie *et al.* [2000] use a compound Poisson process for the jumps with finite Lévy measure (having finitely many big jumps in finite time), whereas Barndorff-Nielsen and Shephard [2001] use a general pure Lévy jump process in $\mathbb{R}$, with infinite Lévy measure (and hence infinitely-many jumps in finite time) that is

empirically reasonable to use for high frequency data. The latter use the five minute return series for the $DM/\$$ while the former the daily return series of the $S\&P$ 500. When pricing options, one must thus pay attention to the frequency of the underlying. The higher the frequency, the more sense it makes to choose a Lévy process having infinite Lévy measure as Barndorff-Nielsen and Shephard [2001], while the lower the frequency the more it makes sense to choose a Lévy process with finite Lévy measure as Duffie *et al.* [2000].

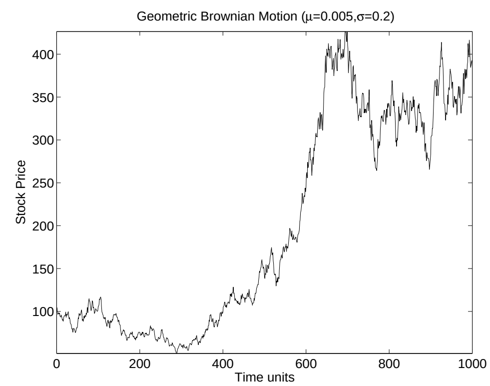
2.3. Example of a Lévy-Driven Stock Process

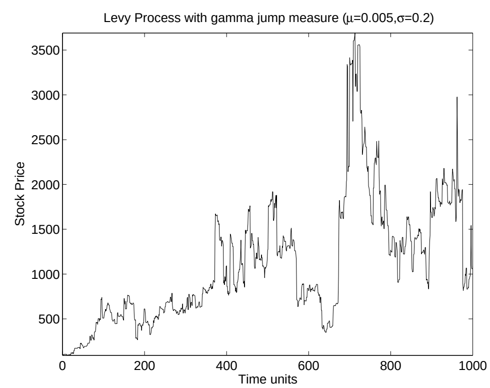
One cannot perhaps fully grasp the impact of introducing small and big jumps has on the price of an underlying S_t. To this end, it is interesting of simulating the sample path of a classical Geometric Brownian motion and to see how the sample paths look like after we introduce small and big jumps. It is often easier to simulate the sample path of a Lévy process by using Brownian subordination, for one can simulate the subordinator first as in Wolpert and Ickstadt [1998], and then simulate Brownian motion on that stochastic time grid.
To this end, we illustrate how different can the sample path from the classical geometric Brownian motion looks with respect to a process driven not only by a diffusion term but a Lévy gamma process which is a subordinator and has an infinite number of arbitrary small jumps. The stochastic differential equations from which we will simulate the sample paths for the stock price S_t are Geometric Brownian motion and subordinated Brownian motion respectively:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t \tag{2.8}$$

$$\frac{dS_t}{S_t} = \mu dU_t + \sigma dW_{(U_t)} \tag{2.9}$$

where $U(t)$ is a subordinator whose jumping Lévy measure is equal to $\nu(dx) = x^{-1}\exp\left(-\frac{x}{\beta}\right)\alpha dx$. W_t is a standard Brownian motion independent of $U(t)$. We have discretized equations 2.8 and 2.9 in order to simulate sample paths with $\mu = 0.005$, $\sigma = 0.2$, $\alpha = 10$, $\beta = 2$ and $S_0 = 100\$$ by using the method of Wolpert and Ickstadt [1998] for simulating the gamma Lévy process U_t combined with the Brownian subordination of Cont and Tankov [2003]. [see Cont and Tankov 2003, p. 183, for simulation of equation 2.9 by Brownian subordination]. The graphs exhibit jumps and larger movements in the process from equation 2.9 than our geometric Brownian motion from 2.8.





3. Risk Neutral Pricing and the "Greeks"

3.1. The Mathematics of Risk Neutral Pricing

As shown in Ait-Sahalia [2004], the essence of good risk management is the ability to disentangle volatility from big and small jumps. Surprisingly, Carr *et al.* [2002] find that pure Lévy jump processes with an infinite number of small jumps perform better at describing return time series than diffusions under the physical measure as well as the risk-neutral one. This has an impact in option pricing, where the integration of such small jumps can lead to better option prices. We now apply the pricing method of Duffie *et al.* [2000] that we extend to stochastic processes described by equation (1.4) that incorporate not only a diffusion term together with big jumps, but a pure Lévy jump process with an infinite number of small jumps.

The density process Z_t (see theorem (2)) is used to make change of measures when we are using Brownian motion. If we use more complicated process, who have càdlàg[6] sample paths, this method is the same, see [Baxter and Rennie 1996, p. 71] for a good example.

To price options, we have to perform a change of measure in (1.4) from the original $\mathbb{P}$ to a risk-neutral $\mathbb{Q}$. The following theorem 2 applies for a general process X_t given by equation (1.4).

[6]"Continu à droite et limites à gauche."

Theorem 2 *Assume* Z *is* well-behaved *at* (b, T), *for some* $b \in \mathbb{R}^n$,
and let $\xi_t = \exp\left(\int_0^t R(X_s)ds + \alpha(t,T,b) + \beta(t,T,b)X_t\right)$ *be the density process, and set* $\frac{d\mathbb{P}}{d\mathbb{Q}} = \frac{\xi_T}{\xi_0}$.
Then $\mathrm{Z}^{\mathbb{Q}}$ *is given by:*

$$\begin{aligned} K_0^{\mathbb{Q}}(t) &= K_0 + H_0\beta(t,T,b) \\ K_1^{\mathbb{Q}}(t) &= K_1 + H_1\beta(t,T,b) \\ H_0^{\mathbb{Q}} &= H_0 \\ H_1^{\mathbb{Q}} &= H_1 \\ \nu_t^{\mathbb{Q}}(dx) &= \exp\left(\beta(t,T,b)x\right)\nu(dx) \end{aligned}$$

We should point out that what is called *Equivalent Martingale Measures* (EMM) in the literature, preserves the original probabilistic structure of the model after the change of measure from $\mathbb{P}$ to $\mathbb{Q}$ (see Nicolato and Venardos [2003]). We start out with a process X_t that is the sum of a Brownian motion and a pure Lévy jump process J_t, but after performing a change of measure, we do not know whether J_t is a pure Lévy jump process, since the measure $\nu_t^{\mathbb{Q}}(dx)$ has a time dependence through $\beta(t,T,b)$, which tells us that J_t is an additive process (see Sato [1999]). If $\beta(t,T,b)$ is independent of time, then we have a Lévy process. Depending on the initial Lévy measure $\nu(dx)$ that we specify, we have to put specific constraints on $\beta(t,T,b)$ to get back the same Lévy measure, albeit with different parameters. This idea is explored very thoroughly in Nicolato and Venardos [2003]. Suppose we can get to $\nu^{\mathbb{Q}}(dx)$ from $\nu(dx)$ given the characteristics of X_t, then it would imply that $\beta(t,T,b)$ is constant in time, thus the quadratic form $\dot{\beta} = \rho_1 - K_1^T\beta - \frac{1}{2}\beta^T H_1\beta$ becomes $0 = \rho_1 - K_1^T b - \frac{1}{2}b^T H_1 b$ (since $\beta(t,T,b) = \beta(T,T,b) = b$) thus leading to a solution, ableit complex-valued for u.

3.2. The Greeks

It is of the utmost importance that once we have call prices either analytically or numerically, we are able to derive the Greeks in order for traders to hedge risk.
The goal is to invert (1.2) over a partition $\mathcal{T} = \{0{=}t_0 < t_1 < \cdots < t_n{=}T\}$ and $X_{\mathcal{T}} = \{X_{t_0}, X_{t_1}, \ldots, X_{t_n}\}$ in order to get a sequence of points used in inverting (1.2). The Greeks are the computed through the following approximations:

$$\Delta \approx \frac{C(S_t + \Delta S, X_{-S}, t) - C(S_t - \Delta S, X_{-S}, t)}{2\Delta S} \tag{3.10}$$

$$\Gamma \approx \frac{C(S_t + \Delta S, X_{-S}, t) - 2C(X,t) + C(S_t - \Delta S, X_{-S}, t)}{\Delta S^2} \tag{3.11}$$

$$\rho \approx \frac{C(r_t + \Delta r, X_{-r}, t) - C(r_t - \Delta r, X_{-r}, t)}{2\Delta r} \tag{3.12}$$

$$\upsilon \approx \frac{C((\sqrt{V} + \Delta\sigma)^2, X_{-V}, t) - C((\sqrt{V} - \Delta\sigma)^2, X_{-V}, t)}{2\Delta\sigma} \tag{3.13}$$

$$\Theta \approx \frac{C(X, t + \Delta t) - C(X, t - \Delta t)}{2\Delta t}. \tag{3.14}$$

where the grid resolution is set with $\Delta\sigma$ as the difference between successive values for the volatility, and e.g. X_{-S} is the vector process excluding the stock price S_t. In our context, X_t is a vector that includes S_t, σ_t, r_t and can incorporate other arbitrary stochastic processes.

4. Conclusion

We have constructed a vector-valued stochastic process that can be used to incorporate many other factors besides the dynamics of the underlying in the pricing of an option. Examples of such factors include stochastic volatility, transaction costs, a stochastic interest rate, and so forth. The introduction of a general Lévy process, together with the pricing framework of Duffie *et al.* [2000], allows us to consider an additional methodology to that of the PIDE of Cont and Tankov [2003], and therefore to price options whose factors are modelled through a regular diffusion, a compound Poisson process and a pure jump Lévy process exhibiting infinitely many jumps in a finite time. Cont and Tankov [2003] further consider the calibration of exponential Lévy models to the observed option prices leading to an inverse problem for the PIDE that is regularized by the use of relative entropy methods. Although addressing this issue is beyond the scope of this paper, it might serve as a fruitful issue for future research. Because one can distinguish statistically between these three types of processes (see Ait-Sahalia [2004]), their inclusion enriches the modelling and allows us to capture the effect of small but discontinuous jumps on the pricing of options.

Now that we have a model that incorporates a wide range of continuous and discontinuous behavior, future work will focus on developing and implementing a Bayesian method to estimate the parameters (drifts, volatilities, etc.) that could be used to improve the accuracy of the hedging ratios used by traders and building credible intervals around those hedging parameters to assess how confident we are about our estimates.

Appendix

Proof for theorem 1

Let $\Psi_t = \exp(V_t)$ with:

$$V_t = -\int_0^t R(X_s)ds + \alpha(t) + \beta(t) \cdot X_t \tag{A.1}$$

where $\cdot$ is the inner-product in $\mathbb{C}^n$.
We also know that $dX_t = \mu(X_t) + \sigma(X_t)dW_t + dJ_t$ where dJ_t is a pure jump Lévy process with $\int_{\mathbb{R}^n}(\|x\|^2 \wedge 1)\nu(dx) < +\infty$
Thus by Itô's formula we get:

$$\begin{aligned}
\Psi_t &= \Psi_0 + \int_0^t \Psi_{s-} dV_s + \frac{1}{2}\int_0^t \exp(V_{s-})\, d[V,V]_s^c + \sum_{0<s\le t}\{\Psi_s - \Psi_{s-} - \Psi'_{s-}\Delta V_s\} \\
\Psi_t &= \Psi_0 + \int_0^t \Psi_{s-}\left[-R(X_s)ds + \dot{\alpha} ds + \dot{\beta}\cdot X_s ds + \beta \cdot dX_s\right] + \frac{1}{2}\int_0^t \Psi_{s-}\beta^T\sigma(X_s)\sigma^T(X_s)\beta ds \\
&\quad + \sum_{0<s\le t}\{\Psi_s - \Psi_{s-} - \Psi'_{s-}\beta\cdot\Delta X_s\} \\
\Psi_t &= \Psi_0 + \int_0^t \Psi_{s-}\left[-R(X_s) + \dot{\alpha} + \dot{\beta}\cdot X_s + \frac{1}{2}\beta^T\sigma(X_s)\sigma^T(X_s)\beta + \beta\cdot\mu(X_s)\right] ds + \int_0^t \Psi_{s-}\beta\cdot dJ_s \\
&\quad + \sum_{0<s\le t}\Psi_{s-}\{\exp(\beta\cdot dJ_s) - 1 - \beta\cdot dJ_s\}
\end{aligned}$$

We will write β for $\beta(t)$ and α for $\alpha(t)$ to simplify the notation.
Now, $\sum_{0<s\le t}\Psi_{s-}\{\exp(\beta\cdot dJ_s) - 1 - \beta\cdot dJ_s\} = \int_{(0,t]\times\mathbb{R}^n}\Psi_{s-}(\exp(x\cdot\beta) - 1 - x\cdot\beta)\,N(ds,dx)$

For ease of notation let us write I for $I(\|x\|\le 1))$. Then,

$$\begin{aligned}
A &= \int_{(0,t]\times\mathbb{R}^n}\Psi_{s-}(\exp(x\cdot\beta) - 1 - x\cdot\beta)\,N(ds,dx) \\
&= \int_{(0,t]\times\mathbb{R}^n}\Psi_{s-}(\exp(x\cdot\beta) - 1 - Ix\cdot\beta + Ix\beta - x\cdot\beta)\,N(ds,dx) \\
&= \int_{(0,t]\times\mathbb{R}^n}\Psi_{s-}(\exp(x\cdot\beta) - 1 - Ix\cdot\beta)\,N(ds,dx) - \int_{(0,t]\times\mathbb{R}}\Psi_{s-}x\cdot\beta(1-I)N(ds,dx)
\end{aligned}$$

Now, $B = \int_0^t \Psi_{s-}\beta\cdot dJ_s = \int_{(0,t]\times\|x\|\le 1}\Psi_{s-}x\cdot\beta\tilde{N} + \int_{(0,t]\times\|x\|>1}\Psi_{s-}x\cdot\beta N(ds,dx)$ by the Lévy-Itô decomposition of the process J_t. We will write $\tilde{N}$ for $(N(ds,dx) - \nu(ds,dx))$ and this is called the compensated poisson random measure.
Since $\int_{(0,t]\times\|x\|\le 1)} x\tilde{N}$ is a $\mathbb{P}$-martingale as well as $\int_{(0,t]\times\|x\|\le 1)}\Psi_{s-}x\cdot\beta\tilde{N}$, so the only term that is not random and that goes into the drift part is
$\int_{(0,t]\times\mathbb{R}}\Psi_{s-}(\exp(x\cdot\beta(s)) - 1 - Ix\cdot\beta)\nu(ds,dx)$.

We thus have for Ψ_t the following expression:

$$\begin{aligned}\Psi_t = \Psi_0 \quad &+ \quad \int_0^t \Psi_{s-}\left[-R(X_s)+\dot{\alpha}+\dot{\beta}\cdot X_s+\frac{1}{2}\beta^T\sigma(X_s)\sigma^T(X_s)\beta+\beta\cdot\mu(X_s)\right]ds \\ &+ \quad A+B\end{aligned}$$

which simplifies to:

$$\begin{aligned}\Psi_t = \Psi_0 \quad &+ \quad \int_0^t \Psi_{s-}\left[-R(X_s)+\dot{\alpha}+\dot{\beta}\cdot X_s+\frac{1}{2}\beta^T\sigma(X_s)\sigma^T(X_s)\beta+\beta\cdot\mu(X_s)\right]ds \\ &+ \quad \int_{(0,t]\times\|x\|\le 1)} \Psi_{s-}x\cdot\beta\tilde{N}+\int_{(0,t]\times\mathbb{R}} \Psi_{s-}(\exp{(x\cdot\beta(s))}-1-Ix\cdot\beta)\tilde{N} \\ &+ \quad \int_{(0,t]\times\mathbb{R}^n} \Psi_{s-}(\exp{(x\cdot\beta(s))}-1-Ix\cdot\beta)\nu(ds,dx)\end{aligned}$$

We need to put $\int_{(0,t]\times\mathbb{R}^n} \Psi_{s-}(\exp{(x\cdot\beta(s))}-1-Ix\cdot\beta)\nu(ds,dx)$ into the drift and since we are dealing with a Lévy process we have $\nu(dt,dx)=dt\nu(dx)$. The drift D is then equal to:

$$\begin{aligned}D \quad &= \quad \Psi_{t-}\left[-R(X_s)+\dot{\alpha}+\dot{\beta}\cdot X_s+\frac{1}{2}\beta^T\sigma(X_s)\sigma^T(X_s)\beta+\beta\cdot\mu(X_s)\right]dt \\ &+ \quad \Psi_{t-}\left[\int_{\mathbb{R}^n}(\exp{(x\cdot\beta)}-1-x\cdot\beta I)\nu(dx)\right]dt\end{aligned}$$

From this equation we get:

$$\begin{aligned}\Psi_t = \Psi_0 \quad &+ \quad \int_0^t \Psi_{s-}\left[-\rho_0+\dot{\alpha}+\frac{1}{2}\beta^T H_0\beta+\beta\cdot K_0+\int_{\mathbb{R}^n}(\exp{(x\cdot\beta)}-1-x\cdot\beta I)\nu(dx)\right]ds \\ &+ \quad \int_0^t \Psi_{s-}X_s^T\left[-\rho_1+\frac{1}{2}\beta^T H_1\beta+\dot{\beta}+K_1^T\beta\right]ds+\int_0^t \Psi_{s-}\beta\cdot\sigma(X_s)dW_s \\ &+ \quad \int_{(0,t]\times\mathbb{R}^n} \Psi_{s-}(\exp{(x\cdot\beta(s))}-1-Ix\cdot\beta)\tilde{N}\end{aligned}$$

Since $\int_0^t \Psi_{s-}\beta\cdot\sigma(X_s)dW_s$ and $\int_{(0,t]\times\mathbb{R}} \Psi_{s-}(\exp{(x\beta(s))}-1-Ix\beta)\tilde{N}$ are $\mathbb{P}$-local-martingales, so if we want Ψ_t to be a $\mathbb{P}$-local-martingale, then we need the following two equations to be true:

$$\dot{\beta} \quad = \quad \rho_1-K_1^T\beta-\frac{1}{2}\beta^T H_1\beta \tag{A.2}$$

$$\dot{\alpha} \quad = \quad \rho_0-K_0^T\beta-\frac{1}{2}\beta^T H_0\beta-\int_{\mathbb{R}^n}(\exp{(x\cdot\beta)}-1-x\cdot\beta I)\nu(dx) \tag{A.3}$$

Proof for theorem 2

The state density process is the same as in Duffie *et al.* [2000] and is equal to:

$$\xi_t \quad = \quad \exp\left(\int_0^t R(X_s)ds + \alpha(t,T,b) + \beta(t,T,b)\cdot X_t\right)$$

where $\alpha(T,T,b) = 0$ and $\beta(T,T,b) = b$ at the terminal date T. $b \in \mathbb{R}^n$ is such that the characteristic Z is well-behaved. We already know that we need to normalize ξ_t by ξ_0 in order to define a new probability measure $d\mathbb{Q}$ as $\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{\xi_T}{\xi_0}$.
The spirit of the proof is similar to that of theorem 1. Denote E_t as the conditional expectation given the filtration $\mathcal{F}_t$.

$$\begin{aligned}
&= E_t^{\mathbb{Q}}\{\exp\left(-\int_0^T R(X_s)ds + u\cdot X_T\right)\}\\
&= \frac{1}{\xi_t}E_t^{\mathbb{P}}\{\exp\left(-\int_0^T R(X_s)ds + u\cdot X_T + \int_0^T R(X_s)ds + \alpha(T,T,b) + \beta(T,T,b)\cdot X_T\right)\}\\
&= \exp\left(-\int_0^t R(X_s)ds - \alpha(t,T,b) - \beta(t,T,b)\cdot X_t\right)E_t^{\mathbb{P}}\{\exp\left((b+u)\cdot X_T\right)\}\\
&= \exp\left(-\int_0^t R(X_s)ds - \alpha(t,T,b) - \beta(t,T,b)\cdot X_t\right)\\
&\quad E_t^{\mathbb{P}}\{\exp\left(-\int_0^T 0ds + \alpha_1(T,T,b+u) + \beta_1(T,T,b+u)\cdot X_T\right)\}
\end{aligned}$$

where, to simplify notation write α_1 for $\alpha(t,T,b{+}u)$, β_1 for $\beta(t,T,b{+}u)$, α for $\alpha(t,T,b)$ and β for $\beta(t,T,b)$.

$$\begin{aligned}
\exp\left(-\int_0^t R(X_s)ds - \alpha - \beta\cdot X_t + \alpha_1 + \beta_1\cdot X_t\right) &= \exp\left(-\int_0^t R(X_s)ds + \alpha_2 + \beta_2\cdot X_t\right)\\
&= \exp\left(M_t\right)\\
&= \Psi_t
\end{aligned}$$

where $\alpha_2 = \alpha(t,T,b+u) - \alpha(t,T,b)$, $\beta_2 = \beta(t,T,b+u) - \beta(t,T,b)$, $\alpha_2(T,T) = 0 - 0 = 0$, $\beta_2(T,T) = b + u - b = u$, $M_t \equiv -\int_0^t R(X_s)ds + \alpha_2 + \beta_2\cdot X_t$ and under $\mathbb{P}$ we have:

$$\begin{aligned}
dM_t &= -R(X_t)dt + \dot{\alpha_2}dt + \dot{\beta_2}\cdot X_t dt + \beta_2\cdot dX_t\\
dX_t &= \mu(X_t)dt + \sigma(X_t)dW_t + dJ_t
\end{aligned}$$

Thus, by Itô's lemma:

$$\begin{aligned}\Psi_t &= \Psi_0 + \int_0^t \Psi_{s-}\left[-R(X_s) + \dot{\alpha_2} + \dot{\beta_2}\cdot X_s + \frac{1}{2}\beta_2^T\sigma_Q^2(X_s^Q)\beta_2 + \beta_2\cdot\mu_Q(X_s)\right]ds \\ &\quad \int_0^t \Psi_{s-}\beta_2\cdot dJ_s + \int_0^t \Psi_{s-}\beta_2\cdot\sigma_Q(X_s)dW_s + \sum_{0<s\leq t}\Psi_{s-}\{\exp(\beta_2\cdot dJ_s) - 1 - \beta_2\cdot dJ_s\}\end{aligned}$$

Our goal is to make Ψ_t a $\mathbb{Q}$-local martingale as we did under $\mathbb{P}$. The parameters under $\mathbb{Q}$ are not the same as they are under $\mathbb{P}$. This is why we put a subscript near each one, and the goal is to recover the current parameters as functions from the previous ones under $\mathbb{P}$. We will derive the coefficients $H_0^{\mathbb{Q}}$, $H_1^{\mathbb{Q}}$, $K_0^{\mathbb{Q}}$, and $K_1^{\mathbb{Q}}$ in the same spirit as in Duffie *et al.* [2000], as well as the Lévy measure of J_t under $\mathbb{Q}$.
Let $C = \sum_{0<s\leq t}\Psi_{s-}\{\exp(\beta_2\cdot dJ_s) - 1 - \beta_2\cdot dJ_s\} = \int_{(0,t]\times\mathbb{R}^n}\Psi_{s-}(\exp(\beta_2\cdot x) - 1 - \beta_2\cdot x)N(ds,dx)$

$$\begin{aligned}C &= \int_{(0,t]\times\mathbb{R}^n}\Psi_{s-}(\exp(\beta_2\cdot x) - 1 - \beta_2\cdot x)N(ds,dx) \\ &= \int_{(0,t]\times\mathbb{R}^n}\Psi_{s-}(\exp(\beta_2\cdot x) - 1 - I\beta_2\cdot x)N(ds,dx) - \int_{(0,t]\times\mathbb{R}^n}\Psi_{s-}\beta_2\cdot x(1-I)N(ds,dx)\end{aligned}$$

set $D = \int_{(0,t]\times\mathbb{R}^n}\Psi_{s-}(\exp(\beta_2\cdot x) - 1 - I\beta_2\cdot x)N(ds,dx)$, and as we saw in last theorem we have:

$$\int_0^t \Psi_{s-}x\cdot\beta_2 dJ_s = \int_{(0,t]\times\|x\|\leq 1}\Psi_{s-}\beta_2\cdot x\tilde{N} + \int_{(0,t]\times\|x\|>1}\Psi_{s-}\beta_2\cdot xN(ds,dx)$$

set $E = \int_{(0,t]\times\|x\|\leq 1}\Psi_{s-}\beta_2\cdot x\tilde{N}$, and we have that:

$$D = \int_{(0,t]\times\mathbb{R}^n}\Psi_{s-}(\exp(\beta_2\cdot x) - 1 - I\beta_2\cdot x)\tilde{N} + \int_{(0,t]\times\mathbb{R}^n}\Psi_{s-}(\exp(\beta_2\cdot x) - 1 - I\beta_2\cdot x)\nu(ds,dx)$$

Now we need to find $\int_{(0,t]\times\mathbb{R}^n}\Psi_{s-}(\exp(\beta_2\cdot x) - 1 - I\beta_2\cdot x)\nu^{\mathbb{Q}}(ds,dx)$ and so we need to compute $E^{\mathbb{Q}}\{\sum_{0<s\leq t}\Psi_{s-}(\exp(\beta_2\cdot dJ_s - 1 - I\beta_2\cdot dJ_s))\}$ but:

$$E^{\mathbb{Q}}\{\sum_{0<s\leq t}\Psi_{s-}(\exp(\beta_2\cdot dJ_s) - 1 - I\beta_2\cdot dJ_s)\} = E^{\mathbb{P}}\{\sum_{0<s\leq t}\xi_s\Psi_{s-}(\exp(\beta_2\cdot dJ_s) - 1 - I\beta_2\cdot dJ_s)\}$$

which is equal to:
$E^{\mathbb{P}}\{\sum_{0<s\leq t}\xi_{s-}\Psi_{s-}(\exp((\beta_2+\beta)\cdot dJ_s) - \exp(\beta\cdot dJ_s) - I\beta_2\cdot dJ_s\exp(\beta\cdot dJ_s)\}$ which equals:

$$
\begin{aligned}
E^{\mathbb{P}}\{\int_{(0,t]\times\mathbb{R}^n}\xi_{s-}\Psi_{s-}(\exp((\beta+\beta_2)\cdot x)-\exp(\beta\cdot x)-Ix\cdot\beta_2\exp(\beta\cdot x))N(ds,dx)\} &= \\
E^{\mathbb{P}}\{\int_{(0,t]\times\mathbb{R}^n}\xi_{s-}\Psi_{s-}(\exp((\beta+\beta_2)\cdot x)-\exp(\beta\cdot x)-Ix\cdot\beta_2\exp(\beta\cdot x))\nu(ds,dx)\} &= \\
E^{\mathbb{P}}\{\int_{(0,t]\times\mathbb{R}^n}\xi_{s-}\Psi_{s-}(\exp(\beta_2\cdot x)-1-Ix\cdot\beta_2)\exp(\beta\cdot x)\,\nu(ds,dx)\}
\end{aligned}
$$

by the predictable projection of the Poisson random measure $N(ds,dx)$ and its compensator $\nu(ds,dx)$ [see Jacod and Shiryaev 1987, p. 71 for more details], from proposition 1.21.
Therefore equating the Lévy measures we get:

$$
\nu^{\mathbb{Q}}(dt,dx) \quad = \quad \exp(\beta\cdot x)\,\nu(dx)dt \tag{A.4}
$$

Notice that if β is independent of time, then J_t is a Lévy process under $\mathbb{Q}$.
Next we find $H_0^{\mathbb{Q}}$, $H_1^{\mathbb{Q}}$, $K_0^{\mathbb{Q}}$, $K_1^{\mathbb{Q}}$ under the risk neutral measure $\mathbb{Q}$. Recall that:

$$
\begin{aligned}
\Psi_t=\Psi_0 &+ \int_0^t\Psi_{s-}\left[\int_{\mathbb{R}^n}\Psi_{s-}(\exp(\beta_2\cdot x)-1-Ix\cdot\beta_2)\exp(\beta\cdot x)\,\nu(dx)\right]ds \\
&+ \int_0^t\Psi_{s-}\left[-\rho_0+\dot{\alpha_2}+\frac{1}{2}\beta_2^TH_0^{\mathbb{Q}}\beta_2+\beta_2\cdot K_0^{\mathbb{Q}}+\right]ds \\
&+ \int_0^t\Psi_{s-}X_s^T\left[-\rho_1+\frac{1}{2}\beta_2^TH_1^{\mathbb{Q}}\beta_2+\dot{\beta_2}+K_1^{T\mathbb{Q}}\beta_2\right]ds+\int_0^t\Psi_{s-}\beta_2\cdot\sigma^{\mathbb{Q}}(X_s)dW_s \\
&+ \int_{(0,t]\times\mathbb{R}^n}\Psi_{s-}(\exp(x\cdot\beta_2)-1-Ix\cdot\beta_2)\tilde{N}
\end{aligned}
$$

We therefore get the equations:

$$
\begin{aligned}
\dot{\alpha_2} &= \rho_0-\int_{\mathbb{R}^n}\Psi_{s-}(\exp(\beta_2\cdot x)-1-Ix\cdot\beta_2)\nu^{\mathbb{Q}}(dx)-\beta_2\cdot K_0^{\mathbb{Q}}-\frac{1}{2}\beta_2^TH_0^{\mathbb{Q}}\beta_2 \\
\dot{\alpha_1} &= \rho_0-\int_{\mathbb{R}^n}\Psi_{s-}(\exp(\beta_2\cdot x)-1-Ix\cdot\beta_2)\nu^{\mathbb{Q}}(dx)+\dot{\alpha}+\beta\cdot K_0^{\mathbb{Q}}-\frac{1}{2}\beta_1^TH_0^{\mathbb{Q}}\beta_1-\beta_1\cdot(K_0^{\mathbb{Q}}-H_0^{\mathbb{Q}}\beta) \\
&\quad +\frac{1}{2}\beta^TH_0^{\mathbb{Q}}\beta
\end{aligned}
$$

identifying the coefficients multiplying β_1 from this last equation with those of:

$$
\dot{\alpha_1} \quad = \quad 0-\beta_1\cdot K_0-\frac{1}{2}\beta_1^TH_0\beta_1-\int_{\mathbb{R}}\Psi_{s-}(\exp(\beta_1x)-1-Ix\beta_1)\nu(dx)
$$

and using the symmetry of H_0 we have that:

$$\begin{aligned} H_0^{\mathbb{Q}} &= H_0 \\ K_0^{\mathbb{Q}} &= H_0^{Q}\beta + K_0 \\ K_0^{\mathbb{Q}} &= H_0\beta + K_0 \end{aligned}$$

Similarily for $\dot{\beta}_1$:

$$\begin{aligned} H_1^{\mathbb{Q}} &= H_1 \\ K_1^{\mathbb{Q}} &= H_1^{\mathbb{Q}}\beta + K_1 \\ K_1^{\mathbb{Q}} &= H_1\beta + K_1 \end{aligned}$$

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