Fractional Ornstein-Uhlenbeck Lévy Processes and the Telecom Process: Upstairs and Downstairs

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Abstract

We model the workload of a network device responding to a random flux of work requests with various intensities and durations in two ways, a conventional univariate stochastic integral approach ("downstairs") and a higher-dimensional random field approach ("upstairs"). The models feature Gaussian, stable, Poisson and, more generally, infinitely divisible distributions reflecting the aggregate work requests from independent sources. We focus on the fractional Ornstein-Uhlenbeck Lévy process and the Telecom process which is the limit of renewal reward processes where both the interrenewal times and the rewards are heavy-tailed. We show that the Telecom process can be interpreted as the workload of a network responding to job requests with stable infinite variance intensities and durations and that fractional Brownian motion can be interpreted in the same way but with finite variance intensities. This explains the ubiquitous presence of fractional Brownian motion in network traffic.

Key words: Fractional Brownian motion, Fractional Lévy motion, Fractional stable motion, Infinitely divisible distributions, Lévy processes, Moving averages, Stable processes.
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1 Introduction: modeling jobs

We wish to model the instantaneous work-load $X'(t)$ and the cumulative work $X(t) = \int_0^t X'(s)\,ds$ of a network device responding to a random incident flux of work requests of varying intensities υ and durations τ, arriving at random times σ. For technical reasons we also construct the weighted load $X[\phi] = \int_{\mathbb{R}} \phi(s)\,X'(s)\,ds$ for suitable weighting functions $\phi(s)$ on $\mathbb{R}$. The models will feature infinitely-divisible distributions for υ to reflect the aggregate work requests from a number of independent sources. We present a *general methodology* for constructing these models and we illustrate it with a number of examples, such as the fractional Ornstein Uhlenbeck process, fractional Brownian motion and the Telecom process.

Let $M(d\sigma\,d\tau)$ be a random measure on $\mathbb{R}\times\mathbb{R}_+$, with a specified infinitely-divisible (ID) distribution. In the sequel we will consider Gaussian, α-stable, gamma, and more generally Lévy random measures in detail. For $t \ge 0$ and suitable functions $\phi(s)$ on $\mathbb{R}$ (again, details to follow) define

$$
\begin{aligned}
X'(t) &= \int_{-\infty}^{\infty}\int_0^{\infty} 1_{(\sigma,\sigma+\tau]}(t)\,M(d\sigma\,d\tau) &\quad (1)\\
X[\phi] &= \int_{-\infty}^{\infty}\int_0^{\infty}\int_{\sigma}^{\sigma+\tau} \phi(s)\,ds\,M(d\sigma\,d\tau) &\quad (2)\\
X(t) &= \int_{-\infty}^{\infty}\int_0^{\infty} |(\sigma,\sigma+\tau]\cap(0,t]|\,M(d\sigma\,d\tau). &\quad (3)
\end{aligned}
$$

where $1_{(\sigma,\sigma+\tau]}(t)$ in Eqn. (1) denotes the indicator function of the interval $A = (\sigma, \sigma+\tau]$ (equal to one if $t \in A$ and zero otherwise), and where the expression $|A|$ in Eqn. (3) denotes the Lebesgue measure (length, in one dimension) of the set A. Eqn. (1) expresses the instantaneous work load $X'(t)$ at time $t \in \mathbb{R}$ as the aggregate sum of all the measure $M(d\sigma\,d\tau)$ arising from work requests beginning at times $\sigma < t$ and lasting until times $\sigma + \tau \ge t$; Eqn. (3) expresses the cumulative work load $X(t)$ from time zero to time t as the integral of $X'(s)$ over that interval; and Eqn. (2) expresses the aggregation of all work, weighted by the function $\phi(s)$. If the indicator function $\phi(s) \equiv 1_{(0,t]}(s)$ is in the domain of $X[\cdot]$, then $X[\phi]$ reduces to $X(t)$; if the Dirac delta function $\phi = \delta_t(s)$ is in the domain of $X[\cdot]$, then $X[\phi]$ reduces to $X'(t)$.

We offer specific examples, and find conditions on the weight function ϕ and the random measure M to ensure that $X'(t)$, $X(t)$, and $X[\phi]$ are well-defined, and show the relationship of these processes to the Telecom process of Levy and Taqqu (2000), Pipiras and Taqqu (2000) and to fractional Brownian motion (Kolmogorov, 1940; Mandelbrot and Van Ness, 1968). Our goal is to offer a general perspective on the various routes one can take in defining models of the types in Eqns. (1–3).

In Section 2 we introduce as motivating examples two representations for the fractional Ornstein-Uhlenbeck and fractional Brownian motion processes— one “downstairs”, driven by Brownian motion, and one “upstairs”, driven by a two-parameter Brownian sheet. In Section 3 we generalize these to a limited class of infinitely divisible processes, again whose two representations now lead to distinct processes. In Section 4 we present a more general framework for moving-average construction and moving-average representations for stationary and stationary-increment processes with infinitely divisible distributions. In Section 5 we consider the Telecom process and several variations, and offer a new interpretation of that process helpful for seeing how it and its variations can be used in telecommunications modeling.

2 Fractional OU Gaussian processes

Let $W = \{W(t)\}_{t\in\mathbb{R}}$ be the standard Wiener process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Fix $\sigma > 0$ and $\lambda > 0$ and for $t \in \mathbb{R}$ define the Wiener integrals

$$Z_t^1 \equiv \sigma\sqrt{2\lambda} \int_{-\infty}^t e^{-\lambda(t-s)}\, W(ds). \tag{4}$$

The process $t \to Z_t^1$ is the familiar Ornstein-Uhlenbeck (or "OU") process, a stationary centered Gaussian stochastic process with covariance function

$$\rho^1(t) = \mathbb{E} Z_0^1 Z_t^1 = \sigma^2\, 2\lambda \int_{-\infty}^{0\wedge t} e^{-\lambda(t-s)} e^{-\lambda(0-s)}\, ds = \sigma^2 e^{-\lambda|t|}. \tag{5}$$

Now construct a series of processes indexed by κ via the recipe

$$Z_t^\kappa = \int_{-\infty}^t \lambda e^{-\lambda(t-s)}\, Z_s^{\kappa-1}\, ds$$

i.e.

$$\begin{aligned}
Z_t^2 &= \int_{-\infty}^t \lambda e^{-\lambda(t-s)}\, Z_s^1\, ds \\
&= \iint_{-\infty<s<s'<t} \lambda e^{-\lambda(t-s')} \sigma\sqrt{2\lambda} e^{-\lambda(s'-s)}\, W(ds)\, ds' \\
&= \sigma\sqrt{2\lambda} \int_{-\infty}^t \lambda(t-s) e^{-\lambda(t-s)}\, W(ds) \\
Z_t^3 &= \int_{-\infty}^t \lambda e^{-\lambda(t-s)}\, Z_s^2\, ds \\
&= \sigma\sqrt{2\lambda} \int_{-\infty}^t \frac{\lambda^2(t-s)^2}{2!} e^{-\lambda(t-s)}\, W(ds)
\end{aligned}$$

and, in general,

$$Z_t^\kappa = \sigma\sqrt{2\lambda} \int_{-\infty}^t \frac{\lambda^{\kappa-1}(t-s)^{\kappa-1}}{\Gamma(\kappa)} e^{-\lambda(t-s)}\, W(ds). \tag{6}$$

For arbitrary $\kappa > \frac{1}{2}$ (not necessarily integral) we take Eqn. (6) to be the definition of Z_t^κ, and call it the *fractional Ornstein-Uhlenbeck* (or "fOU") Gaussian process. Evidently it too is a stationary centered Gaussian process, with covariance function

$$\begin{aligned}
\rho^\kappa(t) &= \mathbb{E} Z_0^\kappa\, Z_t^\kappa \\
&= \sigma^2 e^{-\lambda|t|} \frac{2\lambda^{2\kappa-1}}{\Gamma(\kappa)^2} \int_{-\infty}^0 (|t|-s)^{\kappa-1}(-s)^{\kappa-1} e^{2\lambda s}\, ds \\
&= \frac{2\sigma^2 e^{-\lambda|t|}}{\Gamma(\kappa)^2} \int_0^\infty (\lambda|t|+x)^{\kappa-1} x^{\kappa-1} e^{-2x}\, dx && (7) \\
&= \frac{2\sigma^2}{\Gamma(H-\frac{1}{2})\sqrt{\pi}} \quad (\lambda|t|/2)^{H-1} \quad \mathcal{K}_{H-1}(\lambda|t|) && (8)
\end{aligned}$$

where

$$H \equiv \kappa + \frac{1}{2}$$

and where $\mathcal{K}_\nu(z)$ is the modified Bessel function of the second kind (Abramowitz and Stegun, 1964, §9.6.21–25). It follows from Eqn. (7) that $\rho^\kappa(0) < \infty$ for $\kappa > \frac{1}{2}$ and hence the fOU process is well-defined. Fig. (1) shows plots of $\rho^\kappa(t)$ for $t \in (0, 3)$ and a few values of κ. The function $\rho^\kappa(t)$ falls off like a power $\rho^\kappa(t) \approx \rho^\kappa(0) - c|t|^{2\kappa-1}$ near $t \approx 0$ (see Eqn. (10) below), then like an exponential $\rho^\kappa(t) \approx ce^{-\lambda|t|}$ for large t. Note the contrast with the covariance functions described in Section 4.4.1 below, which act like an exponential function for small t and like a power for large t.

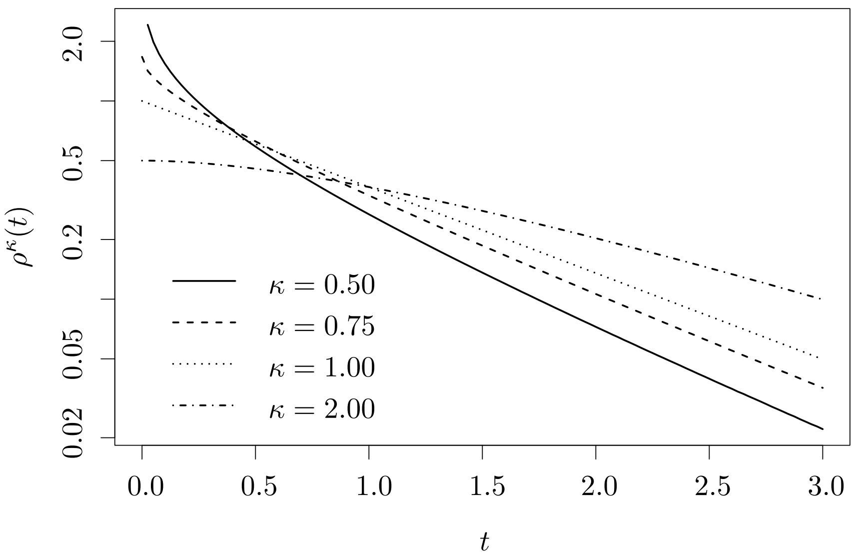

Figure 1: fOU covariance function $\log \rho^\kappa(t)$ on log scale

The spectral density may be shown to be

$$\begin{aligned}
\chi_\kappa(\theta) &\equiv \int_{\mathbb{R}} e^{i\theta t}\, \rho^\kappa(t)\, dt \\
&= \frac{2\sigma^2}{\Gamma(\kappa)\sqrt{\pi}} \int_{\mathbb{R}} \cos(\theta|t|)(\lambda|t|/2)^{\kappa-\frac{1}{2}} \mathcal{K}_{\kappa-\frac{1}{2}}(\lambda|t|)\, dt \\
&= \frac{4\sigma^2}{\lambda\Gamma(\kappa)\sqrt{\pi}} \int_0^\infty \cos(\theta s/\lambda)(s/2)^{\kappa-\frac{1}{2}} \mathcal{K}_{\kappa-\frac{1}{2}}(s)\, ds \\
&= \frac{2\sigma^2}{\lambda}\,(1 + \theta^2/\lambda^2)^{-\kappa}. \qquad (9)
\end{aligned}$$

This is in fact the Matérn (Matérn, 1986) spectral density

$$m_\xi(\theta) = \frac{\xi_1\, {\xi_2}^d}{\Gamma(\xi_3)\pi^{d/2}} \left(1 + {\xi_2}^2\, \theta^2\right)^{-\xi_3 - d/2}$$

in $d = 1$ dimension, with intensity parameter $\xi_1 = 2\sigma^2\sqrt{\pi}\,\Gamma(H-1)$, rate $\xi_2 = \lambda^{-1}$, and shape $\xi_3 = H-1$. Gaussian processes with the isotropic Matérn covariance are commonly used in geostatistical modeling and spatial interpolation or "Kriging" (Cressie, 1993; Chilès and Minières, 1999); now we see that the fOU process is the special case in $d = 1$ dimension.

The necessary and sufficient condition $\kappa > \frac{1}{2}$ for the fOU process to be well-defined also follows from Eqn. (9), which ensures that $\chi_\kappa \in L^1(\mathbb{R}, d\theta)$.

A stochastic process or random field is called *self-similar* if a linear rescaling of the indexing parameter (often "time") by any factor $c > 0$ has the same effect in distribution as a rescaling of the process or field by the factor c^H (*i.e.*, $X(c\,t) \overset{d}{=} c^H X(t)$) for some $H \in \mathbb{R}$; the power H is called the (Hurst) *index* of self-similarity (Hurst, 1951; Samorodnitsky and Taqqu, 1994), and the notation "$\overset{d}{=}$" indicates that two stochastic processes are identical in distribution, *i.e.*, have the same finite-dimensional distributions. Now, from $\mathcal{K}_\nu(z) \sim \frac{1}{2}\Gamma(\nu)(z/2)^{-\nu}$ as $z \to 0$ (Abramowitz and Stegun, 1964, Eqn. 9.6.9) it follows from Eqn. (8) that

$$\rho^\kappa(0) = \frac{\Gamma(H-1)\,\sigma^2}{\Gamma(H-\frac{1}{2})\sqrt{\pi}}$$

and

$$\begin{aligned}
\mathbb{E}\Big[(Z_t^\kappa - Z_0^\kappa)^2\Big] &= 2[\rho^\kappa(0) - \rho^\kappa(t)] \\
&= \frac{4\sigma^2}{\Gamma(\kappa)^2}\int_0^\infty \big[x^{\kappa-1} - (x+\lambda|t|)^{\kappa-1}e^{-\lambda|t|}\big]x^{\kappa-1}e^{-2x}\,dx \\
&= \frac{4\sigma^2(\lambda|t|)^{2\kappa-1}}{\Gamma(\kappa)^2}\int_0^\infty \big[r^{\kappa-1} - (r+1)^{\kappa-1}e^{-\lambda|t|}\big]r^{\kappa-1}e^{-2\lambda|t|r}\,dr \\
&= \frac{4\sigma^2(\lambda|t|)^{2\kappa-1}}{\Gamma(\kappa)^2}\int_0^\infty \big[r^{\kappa-1} - (r+1)^{\kappa-1}\big]r^{\kappa-1}\,dr\,\big(1+O(t)\big) \\
&\sim \frac{-2\sigma^2\sqrt{\pi}}{\Gamma(\kappa)\Gamma(\kappa+\frac{1}{2})\cos(\pi\kappa)}\,(\lambda|t|/2)^{2\kappa-1} \\
&= \frac{-2\sigma^2}{\Gamma(2\kappa)\cos(\pi\kappa)}\,|\lambda t|^{2\kappa-1}
\end{aligned} \tag{10}$$

as $t \to 0$ for $\frac{1}{2} < \kappa < 1$, that is, $1 < H < 3/2$. Eqn. (10) shows that the non-stationary process $Z^\kappa(t) - Z^\kappa(0)$ is asymptotically self-similar with index $(2\kappa-1)/2 = (2H-2)/2 = H - 1 \in (0, \frac{1}{2})$, for small times $t \approx 0$.

What would happen, however, if after rescaling by a power of λ one lets $\lambda \downarrow 0$ in the expression Eqn. (6) defining Z_t^κ? Formally the spectral density $\chi_\kappa(\theta)$ of Eqn. (9) would be proportional to a power of θ, suggesting that the process would then be self-similar. Observe, however, that the process

$$X'(t) \equiv \frac{Z_t^\kappa}{\sqrt{2}\lambda^{\kappa-\frac{1}{2}}} = \frac{\sigma}{\Gamma(\kappa)}\int_{-\infty}^t (t-s)^{\kappa-1}e^{-\lambda(t-s)}\,W(ds) \tag{11}$$

is not well-defined with $\lambda = 0$ for any κ, because $\int_{-\infty}^t (t-s)^{2(\kappa-1)}\,ds = \infty$. There are two ways of overcoming this difficulty. One is to consider, as in Eqn. (2), random fields (sometimes called "generalized random processes") indexed by functions $\phi(\cdot)$; the other is to consider processes with stationary increments (*i.e.*, the random field $X[\phi]$ with $\phi = 1_{(0,t]}$). The latter, as we shall see, is just the well-known fractional Brownian Motion process (Kolmogorov, 1940; Mandelbrot and Van Ness, 1968).

2.1 The random field route

Consider the random field

$$X[\phi] = \frac{\sigma}{\Gamma(\kappa)} \iint_{\mathbb{R}^2} \phi(t)\,(t-s)_+^{\kappa-1}\,dt\,W(ds) \tag{12}$$

for a suitable space Φ of "test functions" $\phi(\cdot)$ (described below). The collection $\{X[\phi]:\ \phi\in\Phi\}$ is called a Gaussian random field— a collection of mean-zero Gaussian random variables $X[\phi]$ indexed, not by time $t\in\mathbb{R}$, but by functions $\phi\in\Phi$. If X'_t of Eqn. (11) were well-defined for $\lambda=0$, one could view $X[\phi]$ as its smoothed version, $X[\phi]=\int_{\mathbb{R}}\phi(t)\,X'(t)\,dt$, a ϕ-weighted average of X'_t. The presence of a power function in Eqn. (12) is characteristic of self-similarity[1].

For two elements $\phi,\psi\in\Phi$ the covariance between $X[\phi]$ and $X[\psi]$ is

$$\operatorname{Cov}\big[X[\phi],X[\psi]\big] = \frac{\sigma^2}{\Gamma(\kappa)^2}\iint_{\mathbb{R}_+^2}\int_{\mathbb{R}}\phi(s+u)\psi(t+u)\,s^{\kappa-1}t^{\kappa-1}du\,ds\,dt. \tag{13}$$

One can also express that covariance in terms of the limiting spectral density $\chi_\kappa(\theta)$ from Eqn. (9), $\lim_{\lambda\to 0}\frac{1}{2}\lambda^{1-2\kappa}\chi_\kappa(\theta)=\sigma^2|\theta|^{-2\kappa}$,

$$\operatorname{Cov}\big[X[\phi],X[\psi]\big] = \frac{\sigma^2}{2\pi}\int_{\mathbb{R}}\hat\phi(\theta)\,\overline{\hat\psi(\theta)}\,|\theta|^{-2\kappa}\,d\theta \tag{14}$$

where the circumflex ("hat") denotes the Fourier transform, and the over-bar complex conjugation.

Eqn. (14) may be used to characterize a suitable family Φ of test functions $\phi(\cdot)$ for which the Gaussian random field $\{X[\phi],\ \phi\in\Phi\}$ will be well-defined. For example, taking $\psi=\phi$, one observes that the variance

$$\mathbb{E}X[\phi]^2 = \frac{\sigma^2}{2\pi}\int_{\mathbb{R}}|\hat\phi(\theta)|^2\ |\theta|^{-2\kappa}\,d\theta \tag{15}$$

is finite for any $\kappa\in(-\infty,M+\frac{1}{2})$ if $|\hat\phi(\theta)|$ decreases faster at infinity than any polynomial and

$$|\hat\phi(\theta)| = O\big(|\theta|^M\big)\ \text{ as } \theta\to 0. \tag{16}$$

Since long-range dependence is characterized by how quickly the spectral density $\chi(\theta)$ blows up at the origin, larger values of M allow one to model greater amounts of long-range dependence in this way. For more details about this approach, see for example Dobrushin (1979); Dobrushin and Major (1979); Major (1981).

2.2 The fractional Brownian motion route

Start with the fOU process of Eqn. (6) and focus on its indefinite integral

$$\int_0^t Z_u^\kappa du = \frac{\sigma\sqrt{2}\lambda^{\kappa-\frac{1}{2}}}{\Gamma(\kappa)}\int_{\mathbb{R}}\int_0^t(u-s)_+^{\kappa-1}e^{-\lambda(u-s)}du\,W(ds), \tag{17}$$

[1] A stationary random field is said to be self-similar with index H if for all $\phi\in\Phi$, $X[\phi]$ and $X[\phi_c]$ have the same probability distribution for all $c>0$, where $\phi_c(t)=c^{-H}\phi(t/c)$ (Major, 1981, $p.$ 5). Since $\hat\phi_c(\theta)=c^{-H+1}\,\hat\phi(\theta c)$, it follows from Eqn. (14) that the Gaussian random field $X[\phi]$ is self-similar with index $H=\kappa+\frac{1}{2}$.

a well-defined centered Gaussian process for $\kappa > -\frac{1}{2}$ with stationary increments. Now rescale it (dividing by $\sqrt{2}\lambda^{\kappa-\frac{1}{2}}$) and take $\lambda \searrow 0$, leading to a process

$$X(t) \equiv \frac{\sigma}{\Gamma(\kappa+1)} \int_{\mathbb{R}} \Big[(t-s)_+^{\kappa} - (-s)_+^{\kappa}\Big] W(ds) \tag{18}$$

known as *fractional Brownian motion* (fBM). Well defined in the range $-\frac{1}{2} < \kappa < \frac{1}{2}$, that is, $0 < H < 1$ for $H = \kappa + \frac{1}{2}$, it has mean zero, variance $\mathbb{E}X(t)^2 = c|t|^H$, for some $c > 0$, and covariance

$$\mathbb{E}\, X(s)\, X(t) = \frac{c}{2}\left\{|s|^{2H} + |t|^{2H} - |t-s|^{2H}\right\} \tag{19}$$

(see Taqqu, 2003, for details).

The fBM $X(t)$ can also be expressed in terms of random fields. To see this, first note that its derivative $X'(t)$ is not defined as a stochastic process, but if it were, it would have a representation of the form Eqn. (11); and that to obtain fBM we integrated fOU in Eqn. (17), and then rescaled and took $\lambda \to 0$. Thus the fBM $X(t)$ is (up to a scaling constant) the process $X[\phi_t]$ in Eqn. (12) with $\phi_t(u) = 1_{(0,t]}(u)$ for $t \geq 0$. Since the Fourier transform of this test function is $\hat{\phi}_t(\theta) = \big(e^{i\theta t}-1\big)/(i\,\theta)$, one can verify that the variance given in Eqn. (15) is finite (and hence $X(t)$ well-defined) for $-\frac{1}{2} < \kappa < \frac{1}{2}$. Observe that Eqn. (16) is satisfied here with $M = 0$, leading to the restriction $\kappa < \frac{1}{2}$; the further restriction $\kappa > -\frac{1}{2}$ arises because $|\hat{\phi}_t(\theta)| \sim |\theta|^{-1}$ as $\theta \to \pm\infty$.

This illustrates the relationship introduced in Section 1 between the three processes $X'(t)$, $X[\phi_t]$, and $X(t)$. The representations used here (for example, Eqn. (18)) for the fBM $X(t)$ differ from those of Eqns. (1–3), but we will see below, however, that the representation Eqn. (1) is also valid for fBM. There is therefore a close analogy between the processes $X'(t)$, $X[\phi_t]$, and $X(t)$ and those of Eqns. (1–3).

2.3 The "upstairs" representation

We now introduce a new way of constructing the fOU and related processes, which we call the "upstairs" representation, as the random measure of a moving geometric figure in a higher-dimensional space ($\mathbb{R}^2$, in the example) or, equivalently, as the stochastic integral of a 0/1-valued function. In Section 3.3.1 we will present a networking interpretation of this representation, making it easy to see how to tailor it to specific networking applications, with network work requests following either Gaussian or general infinitely divisible distributions. First we consider the Gaussian case.

We will show that any stationary Gaussian process Z_t with differentiable covariance function $\rho(t)$ that is non-increasing and convex on $\mathbb{R}_+$ can be written in the form of a two-dimensional stochastic integral

$$Z_t = c \int_{\mathbb{R}} \int_0^{g(t-x_2)} W(dx_1\, dx_2) \tag{20}$$

where $c > 0$ is arbitrary,

$$g(t) \equiv -c^{-2}\rho'(t) 1_{\mathbb{R}_+}(t),$$

and where $W(x_1, x_2)$ denotes the *Brownian sheet*, the two-parameter continuous-path centered Gaussian process with covariance function $\mathbb{E}W(x_1,x_2)W(y_1,y_2) = (|x_1| \wedge |y_1|)(|x_2| \wedge |y_2|)$ for $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$ with $x_1\, y_1 > 0$ and $x_2\, y_2 > 0$, otherwise zero (see, for example, Orey and Pruitt, 1973; Walsh, 1986, for more details about Brownian sheets). Evidently Z_t from Eqn. (20)

is Gaussian with zero mean, since $W(x_1, x_2)$ is. Since $g(t)$ is decreasing on $\mathbb{R}_+$ (by the convexity of $\rho(t)$), the covariance is:

$$
\begin{aligned}
\mathbb{E} Z_0 Z_t &= c^2 \int_{-\infty}^{0\wedge t} \int_0^{g(t-x_2)\wedge g(0-x_2)} dx_1\, dx_2 \\
&= c^2 \int_{-\infty}^{0} g(|t|-x)\, dx \\
&= c^2 \int_{|t|}^{\infty} g(x)\, dx \\
&= \rho(t)
\end{aligned} \tag{21}
$$

since $g(x) = -c^{-2}\rho'(x)$ for $x > 0$; thus Z_t has mean zero and covariance $\rho(t)$, as claimed. In particular the OU process Z_t^1 of Eqn. (4) with covariance function $\rho^1(t) = \sigma^2 \exp(-\lambda|t|)$ satisfies this requirement with $c = \sigma\sqrt{\lambda}$ and $g(t) = \exp(-\lambda|t|)$ for $t > 0$, giving the alternate representation:

$$
\begin{aligned}
Z_t^1 &= \sigma\sqrt{\lambda} \int_{\mathbb{R}} \int_0^{g(t-x_2)} W(dx_1, dx_2) \\
&= \sigma\sqrt{\lambda} \int_{-\infty}^{t} \int_0^{e^{-\lambda(t-x_2)}} W(dx_1, dx_2).
\end{aligned} \tag{22}
$$

We refer to Eqn. (22) as the "upstairs" representation and to Eqn. (4) as the "downstairs" representation of the Gaussian OU process Z_t^1. These representations, as we will see in Section 3.3, are no longer equivalent if one replaces the Gaussian measure by a more general infinitely divisible one.

3 Fractional OU Lévy processes

We now consider how to construct a process similar to fOU but driven, not by a Wiener measure, but by a an infinitely divisible (or "ID") measure without a Gaussian component; the resulting process will be called a *Fractional Ornstein-Uhlenbeck Lévy* (or "fOUL") process.

3.1 Construction for ID non-Gaussian processes

Fix $\lambda > 0$ and a nonnegative measure $\nu(du)$ on $\mathbb{R}$ satisfying (for now) the integrability condition

$$
\int_{\mathbb{R}} \big(|u| \wedge 1\big)\, \nu(du) < \infty \tag{23}
$$

where, as usual, $(a \wedge b) \equiv \min(a, b)$ denotes the minimum of two real numbers a, b. Condition (23) is used to simplify the exposition, and will be relaxed later. Let M be the random measure which assigns independent random variables $M(A_j)$ to disjoint sets $A_j \in \mathbb{R}$ with log characteristic function

$$
\log \mathbb{E}\big[e^{i\theta M(A_j)}\big] = |A_j| \int_{\mathbb{R}} \big(e^{i\theta u} - 1\big)\, \nu(du), \tag{24}
$$

where $|A_j|$ denotes the Lebesgue measure of the set $|A_j|$. In view of Eqn. (24) (and the Lévy-Khinchine formula, see Khinchine and Lévy 1936 or *p.* 74 of Rogers and Williams 1994) the random variable $M(A_j)$ has an infinitely divisible (ID) distribution. The measure $\nu(du)$ is called the Lévy measure (in honor of the pioneering work of Lévy (1924)). The process $\zeta_t = M((0, t])$, $t \geq 0$, with stationary independent increments, is called a Lévy process (Lévy, 1924, 1925; Rogers and Williams, 1994, I.28).

The integrability condition Eqn. (23) ensures that the characteristic function in Eqn. (24) is well-defined and that ζ_t has (almost-surely) finite total variation over bounded intervals (Cont and Tankov, 2004, Prop. 3.9), which makes it straightforward to define stochastic integrals $\int_{\mathbb{R}} f(s)\, M(ds)$ for bounded compactly-supported functions $f(s)$ or, more generally, for any function $f(s)$ with

$$\iint_{\mathbb{R}^2} \big(|u\, f(s)| \wedge 1\big)\, \nu(du)\, ds < \infty. \tag{25}$$

Wolfe (1982) and Sato (1999, Theorem 17.5)) showed that $\int_{\mathbb{R}} f(s)\, M(ds)$ is well-defined for functions of the form $f(s) = c\exp(-\lambda s)1_{[0,\infty)}(s)$ for $c, \lambda > 0$, for any $\nu(du)$ that satisfies Eqn. (23) along with

$$\int_{\mathbb{R}} \log_+ |u|\, \nu(du) < \infty, \tag{26}$$

where $\log_+(x) \equiv \log(x)$ for $x > 1$, $\log_+(x) \equiv 0$ for $x \leq 1$. In Appendix B we verify that Eqn. (25) is satisfied in this case, and also, more generally, for any function of the form $f(s) \equiv c\, s^{\kappa-1} \exp(-\lambda s)1_{[0,\infty)}(s)$ with $c, \kappa, \lambda > 0$, if ν satisfies Eqn. (26). This allows us to define processes:

$$\begin{aligned}
Y_t^1 &\equiv \sqrt{2\lambda} \int_{-\infty}^{t} e^{-\lambda(t-s)}\, M(ds) \\
Y_t^2 &\equiv \int_{-\infty}^{t} \lambda e^{-\lambda(t-s)}\, Y_s^1\, ds \\
&= \iint_{-\infty<s<s'<t} \lambda e^{-\lambda(t-s')} \sqrt{2\lambda} e^{-\lambda(s'-s)}\, M(ds)\, ds' \\
&= \sqrt{2\lambda} \int_{-\infty}^{t} \lambda(t-s) e^{-\lambda(t-s)}\, M(ds)
\end{aligned}$$

and, in general,

$$\begin{aligned}
Y_t^\kappa &\equiv \int_{-\infty}^{t} \lambda e^{-\lambda(t-s)}\, Y_s^{\kappa-1}\, ds \\
&= \sqrt{2\lambda} \int_{-\infty}^{t} \frac{\lambda^{\kappa-1}(t-s)^{\kappa-1}}{\Gamma(\kappa)} e^{-\lambda(t-s)}\, M(ds).
\end{aligned} \tag{27}$$

for all $\kappa > 0$. The process Y_t^1 is the Ornstein-Uhlenbeck Lévy (OUL) process and Y_t^κ the fractional Ornstein-Uhlenbeck Lévy (fOUL) process.

Each of these stationary processes has non-anticipating moving-average form

$$Y_t = \int_{\mathbb{R}} h(t-s)\, M(ds) \tag{28}$$

for some function satisfying $h(t) \equiv 0$ for $t < 0$, with $f(s) \equiv h(t-s)$ satisfying Eqn. (25) for each fixed $t \in \mathbb{R}$, and has log characteristic function

$$\begin{aligned}
\log \mathbb{E}[\exp\{i\theta Y_t\}] &= \iint_{\mathbb{R}^2} \Big(e^{i\theta u h(t-s)} - 1\Big)\, \nu(du)\, ds \\
&= \int_0^\infty \int_{\mathbb{R}} \Big(e^{i\theta u h(s)} - 1\Big)\, \nu(du)\, ds \\
&= \int_{\mathbb{R}} \big(e^{i\theta v} - 1\big)\, \tilde{\nu}(dv)
\end{aligned} \tag{29}$$

where, heuristically,

$$\tilde{\nu}(dv) = \int_0^\infty \nu\big(dv/h(s)\big)\,ds$$

or, more precisely, where the measure $\tilde{\nu}$ is characterized by the relation

$$\int_{\mathbb{R}} \phi(v)\tilde{\nu}(dv) = \int_0^\infty \int_{\mathbb{R}} \phi\big(u\,h(s)\big)\nu(du)\,ds \tag{30}$$

for all infinitely differentiable compactly supported functions $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$, so the process Y_t of Eqn. (28) is an ID process with Lévy measure $\tilde{\nu}$. Note $\tilde{\nu}$ also satisfies Eqn. (23), with

$$\int_{\mathbb{R}} \big(|u| \wedge 1\big)\,\tilde{\nu}(du) \equiv \iint_{\mathbb{R}^2} \big(|u\,h(s)| \wedge 1\big)\nu(du)\,ds < \infty$$

by Eqn. (25).

Applying this result to the fOUL process Y_t^κ, we find the log characteristic function of this stationary process is given by Eqn. (29) with Lévy measure $\tilde{\nu}_\kappa$, given (after a change of variables to $x = \lambda s$) by the relation

$$\int_{\mathbb{R}} \phi(u)\tilde{\nu}_\kappa(du) = \int_0^\infty \int_{\mathbb{R}} \phi\big(ux^{\kappa-1}e^{-x}\sqrt{2\lambda}/\Gamma(\kappa)\big)\,\nu(du)\,dx/\lambda \tag{31}$$

for all $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$. In the case of the Ornstein-Uhlenbeck Lévy (OUL) process Y_t^1, with $\phi(u) = (e^{i\theta u} - 1)$, this gives the log characteristic function of Y_t^1 as

$$\begin{aligned} \log \mathbb{E}\big[e^{i\theta Y_t^1}\big] &= \int_{\mathbb{R}} \big(e^{i\theta u} - 1\big)\,\tilde{\nu}(du) \\ &= \int_0^\infty \int_{\mathbb{R}} \Big[\exp\Big(i\,\theta u\,e^{-x}\sqrt{2\lambda}\Big) - 1\Big]\;\nu(du)\,dx/\lambda. \end{aligned} \tag{32}$$

In the finite-variance case, *i.e.*, when $\sigma^2 \equiv \int u^2\nu(du) < \infty$, then the covariance function of any process Y_t of moving-average form Eqn. (28) is given by

$$\begin{aligned} \rho(t) &= \operatorname{Cov}\big(Y_0, Y_t\big) \\ &= \int_{-\infty}^{0\wedge t} \int_{\mathbb{R}} u^2\,h(t-s)\,h(-s)\nu(du)\,ds \\ &= \sigma^2 \int_0^\infty h(|t| + x)\,h(x)\,dx \end{aligned}$$

and, in particular, the covariance function for Y_t^1 is again

$$\rho^1(t) = \sigma^2 e^{-\lambda|t|}$$

and, more generally, for Y_t^κ is

$$\begin{aligned} \rho^\kappa(t) &= \frac{2\sigma^2\,e^{-\lambda|t|}}{\Gamma(\kappa)^2} \int_0^\infty (\lambda|t| + x)^{\kappa-1}(x)^{\kappa-1}e^{-2\,x}\,dx \\ &= \frac{2\sigma^2}{\Gamma(H - \frac{1}{2})\sqrt{\pi}}\quad (\lambda|t|/2)^{H-1}\;\;\mathcal{K}_{H-1}(\lambda|t|), \end{aligned}$$

just as that given in Eqn. (8) for the fOU processes.

3.2 Using a Poisson representation

On any σ-finite measure space $\big(\mathcal{X},\mathcal{F},m(dx)\big)$ one can construct Poisson random measure $N(dx)\sim \mathsf{Po}(m(dx))$ that assigns independent Poisson random variables $N(A_j)$ with means $\mathbb{E}N(A_j)=m(A_j)$ to disjoint sets $A_j\in\mathcal{F}$ of finite m-measure. The Lévy process ζ_t of Section 3.1 can be represented in the form

$$\zeta_t \equiv \int_0^t\int_{\mathbb{R}} u\,N(du\,ds),\quad t\ge 0;\qquad \zeta_t\equiv -\int_t^0\int_{\mathbb{R}} u\,N(du\,ds),\quad t<0$$

for a Poisson random measure $N(du\,ds)\sim\mathsf{Po}\big(\nu(du)\,ds\big)$ on $\mathbb{R}^2$. Intuitively, $N(du\,ds)$ describes the number of points in $du\times ds$ and, if u is interpreted as "height", then ζ_t describes the cumulative height of the points in $\mathbb{R}\times(0,t]$. This gives every moving-average processes of the form Eqn. (28) the more intuitive form

$$Y_t\equiv\iint_{\mathbb{R}^2} h(t-s)\,u\,N(du\,ds). \tag{28$'$}$$

Thus the fOUL process of Eqn. (27) can be represented as

$$Y_t^\kappa\equiv\frac{\lambda^{\kappa-1}\sqrt{2\lambda}}{\Gamma(\kappa)}\int_{-\infty}^t\int_{\mathbb{R}}(t-s)^{\kappa-1}e^{-\lambda(t-s)}\,u\,N(du\,ds) \tag{27$'$}$$

for all $\kappa>\frac{1}{2}$ and in particular

$$Y_t^1\equiv\sqrt{2\lambda}\int_{-\infty}^t\int_{\mathbb{R}} e^{-\lambda(t-s)}\,u\,N(du\,ds) \tag{33}$$

for the case of $\kappa=1$, the Ornstein-Uhlenbeck Lévy (OUL) process.

It was noted in Section 2.3 that, in the Gaussian case of Eqn. (6) for $\kappa=1$, there was no difference between treating the function $h(t-s)\equiv\sigma\sqrt{2\lambda}\,e^{-\lambda(t-s)}$ as an integrand for Brownian motion $W(s)$ (the "downstairs" representation of a moving-average Gaussian process) or considering the "upstairs" representation of Eqn. (20), integrating a two-parameter Brownian sheet $W(r,s)$ over a the two-dimensional set of the form $\{(r,s):\ 0<r\le g(t-s)\}$, for a suitable function $g(x)$ vanishing for $x\le 0$. These differ for Lévy processes, as we will see in the next subsection.

3.3 "Upstairs" Poisson representations

Let $g(t):\mathbb{R}_+\to\mathbb{R}_+$ be nonnegative nonincreasing with finite integral

$$\mu\equiv\int_0^\infty g(x)\,dx<\infty, \tag{34}$$

introduce a Poisson random measure $N(du\,dx_1\,dx_2)\sim\mathsf{Po}(\nu(du)\,dx_1\,dx_2)$ on $\mathbb{R}^3$, and set

$$\begin{aligned} X_t &\equiv \int_{-\infty}^t\int_0^{g(t-x_2)}\int_{\mathbb{R}} u\,N(du\,dx_1\,dx_2) \\ &= \iiint_{\mathbb{R}^3} 1_{\{0<x_1\le g(t-x_2)\}}\,u\,N(du\,dx_1\,dx_2). \end{aligned} \tag{35}$$

The process $\{X_t,\ t\in\mathbb{R}\}$ is stationary; its infinitely-divisible marginal distribution has log characteristic function

$$\log\mathbb{E}\big[e^{i\theta X_t}\big]=\int_{\mathbb{R}}\big(e^{i\theta u}-1\big)\,\mu\,\nu(du) \tag{36}$$

since $\int_{-\infty}^{t}\int_{0}^{g(t-x_2)} dx_1\,dx_2 = \mu$. The process $\{X_t,\ t\in\mathbb{R}\}$ then has Lévy measure $\mu\nu(du)$ and, if $\sigma^2 \equiv \int_{\mathbb{R}} u^2\nu(du) < \infty$, has covariance function

$$\rho(t) \;=\; \mathrm{Cov}\big(X_0, X_t\big) = \sigma^2 \int_{|t|}^{\infty} g(x)\,dx \tag{37}$$

so

$$g(t) \;=\; -\sigma^{-2}\,\frac{d}{dt}\rho(t), \qquad t>0, \tag{38}$$

and $g(t)$ is simply the negative derivative of the correlation function as in the Gaussian case of Eqn. (21).

In particular, for $g(t) = e^{-\lambda t}1_{\{t\ge 0\}}$, then $\mu = \int_{\mathbb{R}_+} g(t)\,dt = \lambda^{-1}$ and the covariance is again given by $\rho(t) = \rho^1(t) = \sigma^2\,e^{-\lambda|t|}$ of Eqn. (5), so of course the spectral density remains $\chi(\theta) = \frac{2\sigma^2}{\lambda}\,(1+\theta^2/\lambda^2)^{-1}$ as in Eqn. (9). In contrast to the Gaussian case, the process X_t with "upstairs" representation Eqn. (35) and the process Y_t^1 with "downstairs" representation Eqn. (33) have different probability distributions (easily seen by comparing log characteristic functions Eqns. (32, 36)), even though they share the same covariance function.

3.3.1 Networking interpretation of the upstairs representation

The process $\{X_t,\ t\in\mathbb{R}\}$ of Eqn. (35) evolves solely by taking positive and negative jumps of same magnitudes. If (u, x_1, x_2) is in the random support of $N(du\,dx_1\,dx_2)$, with $x_1 \in (0, g(0)]$, then $\{X_t,\ t\in\mathbb{R}\}$ will take a jump of magnitude $[X_t - X_{t-}] = u$ at time $t = x_2$. After a lag of length $\tau = g^{-1}(x_1)$, at time $t' = t+\tau$, the indicator function in Eqn. (35) returns to zero and the process $\{X_t,\ t\in\mathbb{R}\}$ takes a return jump of magnitude $[X_{t'} - X_{t'-}] = -u$. In the common case where $\nu(du)$ is supported on $\mathbb{R}_+$, for example, positive jumps of magnitude $u \ge \epsilon$ arrive as a Poisson process with rate $\lambda_\epsilon = \nu\big([\epsilon,\infty)\big) < \infty$; each such jump is followed after a delay τ by a restoring negative jump of magnitude $-u$. Conditional on $x_1 \in (0, g(0)]$, then x_1 is distributed uniformly in that interval and hence the delay τ has a probability distribution with complementary CDF $\mathbb{P}[\tau > t] = \mathbb{P}[g^{-1}(x_1) > t] = \mathbb{P}[x_1 < g(t)] = g(t)/g(0)$; the integrability condition Eqn. (34) now asserts that τ has finite mean $\mathbb{E}\,\tau = g(0)^{-1}\int_0^\infty g(t)\,dt < \infty$. For the case of $g(t) = e^{-\lambda t}$ on $\mathbb{R}_+$, τ has an exponential distribution with mean $\mathbb{E}[\tau] = \lambda^{-1} < \infty$.

In the context of network traffic, the interpretation is that packets (or files) of size $u \ge \epsilon > 0$ are transferred through a given node at Poisson rate $\lambda_\epsilon = \nu\big([\epsilon,\infty)\big) < \infty$, and occupy a node or link for a random duration τ with CDF $\Pr[\tau \le t] = 1 - g(t)/g(0)$. We shall return to this point in Section 4.4.1.

4 The general infinitely divisible case

The ID processes constructed in Section 3.1 were defined under the restrictive condition of Eqn. (23). We relax this restriction in this section and consider integrals with respect to a general Lévy process (corresponding to a random measure M) with both Gaussian and Poisson components.

4.1 Integral representation

We start by recalling the definition of the integral

$$\begin{aligned} M[f] &= \int_{\mathcal{X}} f(x)\, M(dx) && (39)\\ &= \int_{\mathcal{X}} f(x)\, m(dx) + \int_{\mathcal{X}} f(x)\, W(dx) \\ &\quad + \iint_{|uf(x)|\le 1} u\, f(x)\, \tilde{N}(du\, dx) + \iint_{|uf(x)|>1} u\, f(x)\, N(du\, dx) && (40)\end{aligned}$$

where $\mathcal{X}$ is the Euclidean space $\mathbb{R}$ or $\mathbb{R}^2$ (the same construction will work with any complete separable metric space); $f : \mathcal{X} \to \mathbb{C}$ is a real or complex-valued Borel function on $\mathcal{X}$; $m(dx)$ is a real-valued Borel measure on $\mathcal{X}$ (with total variation measure $|m|(dx)$); $W(dx)$ is a zero-mean Gaussian random measure on $\mathcal{X}$ with control (or "mean") measure $v(dx)$ (so $\mathbb{E}W(dx)^2 = v(dx)$); $N(du\, dx)$ is a Poisson random measure on $\mathbb{R} \times \mathcal{X}$ with control measure $\nu(du\, dx)$ (so $\mathbb{E}N(du\, dx) = \nu(du\, dx)$), and $\tilde{N}(du\, dx) = N(du\, dx) - \nu(du\, dx)$ is the corresponding "compensated" random measure[2].

The distribution of the random variable $M[f]$ of Eqn. (39) can be specified by the characteristic function:

$$\begin{aligned} \log \mathbb{E} \exp\{i\,\theta\, M[f]\} &= i\,\theta \int_{\mathcal{X}} f(x)\, m(dx) - \frac{\theta^2}{2} \int_{\mathcal{X}} |f(x)|^2\, v(dx) \\ &\quad + \iint_{\{|u\, f(x)|\le 1\}} \left[e^{i\,\theta\, u\, f(x)} - 1 - i\,\theta u\, f(x) \right] \nu(du\, dx) \\ &\quad + \iint_{\{|u\, f(x)|>1\}} \left[e^{i\,\theta\, u\, f(x)} - 1 \right] \nu(du\, dx). && (41)\end{aligned}$$

This (and hence $M[f]$ in Eqn. (40)) is well-defined under the condition that

$$\int_{\mathcal{X}} |f(x)|\, |m|(dx) \quad < \quad \infty, \qquad \int_{\mathcal{X}} |f(x)|^2\, v(dx) \quad < \quad \infty, \tag{42}$$

and

$$\int_{\mathcal{X}} \int_{\mathbb{R}} \left(|u\, f(x)|^2 \wedge 1 \right) \nu(du\, dx) \quad < \quad \infty. \tag{43}$$

The last condition follows from the fact that the modulus of the kernels in the last two integrals in Eqn. (41) are bounded respectively by $\left|u\, f(x)\right|^2$ and by 2. The collection Φ of functions f satisfying Eqns. (42, 43) is a linear space, since $(|u(f+g)|^2 \wedge 1) \le 2(|u\, f|^2 \wedge 1) + 2(|u\, g|^2 \wedge 1)$. It contains, for example, every bounded measurable compactly-supported function $f : \mathcal{X} \to \mathbb{C}$ if $\nu(du\, dx)$ satisfies the integrability condition

$$\int_{\mathbb{R}} \left(u^2 \wedge 1 \right) \nu(du, K) < \infty \tag{44}$$

for each compact set $K \subset \mathcal{X}$; see Musielak (1983) for a general description of what are now called Musielak-Orlicz spaces.

A sufficient condition for the random variable $M[f]$ of Eqn. (40) to have a finite mean is that Eqn. (42) hold and, in addition,

$$\iint_{\mathbb{R}\times\mathcal{X}} \left(|u\, f(x)|^2 \wedge |u\, f(x)| \right) \nu(du\, dx) < \infty, \tag{45}$$

[2]If $\mathcal{B}$ and $\mathcal{F}$ denote the Borel σ-fields on $\mathbb{R}$ and $\mathcal{X}$, respectively, then $v(dx)$ and $\nu(du\, dx)$ are non-negative sigma-finite measures on $(\mathcal{X}, \mathcal{F})$ and $(\mathbb{R} \times \mathcal{X}, \mathcal{B} \otimes \mathcal{F})$, respectively. The measure ν is also referred to as the Lévy measure.

in which case

$$\mathbb{E}M[f] = \int_{\mathcal{X}} f(x)\, m(dx) + \iint_{\{|u\,f(x)|>1\}} u\, f(x)\, \nu(du\, dx). \tag{46}$$

A necessary and sufficient condition for $M[f]$ to have a finite variance is that Eqn. (42) hold and, in addition,

$$\mathrm{Var} M[f] = \int_{\mathcal{X}} f(x)^2\, v(dx) + \iint_{\mathbb{R}\times\mathcal{X}} u^2\, f(x)^2\, \nu(du\, dx) < \infty. \tag{47}$$

Since $\big(|u\, f(x)|^2 \wedge |u\, f(x)|\big) \le (u^2 \wedge |u|) \times \big(|f(x)|^2 \vee |f(x)|\big)$, Eqn. (45) is satisfied if, as we will often suppose,

$$f \in L^1 \cap L^2(\mathcal{X}, dx) \qquad \nu(du\, dx) = \nu(du)\, dx \qquad \int_{\mathbb{R}} \big(u^2 \wedge |u|\big)\, \nu(du) < \infty. \tag{48}$$

There are also situations where it is not necessary to compensate the Poisson random measure. In this case the integral $M[f]$ may be defined as

$$M[f] \equiv \int_{\mathcal{X}} f(x)\, m_0(dx) + \int_{\mathcal{X}} f(x)\, W(dx) + \iint_{\mathcal{X}\times\mathbb{R}} u\, f(x)\, N(du\, dx) \tag{49}$$

if Eqn. (42) holds with m replaced by m_0 and if Eqn. (43) is replaced by the sharper condition

$$\int_{\mathcal{X}} \int_{\mathbb{R}} \big(|u\, f(x)| \wedge 1\big)\, \nu(du\, dx) < \infty. \tag{50}$$

Eqn. (28), for example, has the representation of Eqn. (49) with $m_0 = 0$, $v(dx) = 0$, and $f(x) = h(t-x)$. The Telecom example considered in Section 5 below requires the weaker restriction of Eqn. (44).

Both Eqns. (40, 49) are special cases of the more general definition

$$\begin{aligned} M_h[f] \;\equiv\; & \int_{\mathcal{X}} f(x)\, m_h(dx) + \int_{\mathcal{X}} f(x)\, W(dx) + \\ & + \iint_{\mathcal{X}\times\mathbb{R}} h\big(u\, f(x)\big)\, \tilde{N}(du\, dx) + \iint_{\mathcal{X}\times\mathbb{R}} \big[u\, f(x) - h\big(u\, f(x)\big)\big]\, N(du\, dx) \end{aligned}$$

for arbitrary bounded *compensation function* $h(x) = x + o(x)$ as $x \to 0$ (see Sato, 1999, *p.* 38)[3] such as $h(x) = \sin x$, $\arctan x$, or $x/(x + x^2)$; in Eqn. (40) we made the conventional choice of $h(x) \equiv x\, 1_{|x|\le 1}$, while in Eqn. (49) we took $h = 0$. The two expressions of Eqns. (40, 49) will coincide if

$$m_0(dx) = m(dx) - \int_{|u\, f(x)|\le 1} u\, \nu(du\, dx);$$

the only effect of changing the compensation function h is to shift the measure $m(dx)$. Eqn. (49) is simpler where it is available (*i.e.*, when Eqn. (50) holds)— this expression for $M[f]$ is linear in

[3] The log characteristic function of $M_h[f]$ has the convenient form

$$\log \mathbb{E} \exp\{i\theta M_h[f]\} = i\theta \int_{\mathcal{X}} f(x) m(dx) - \frac{\theta^2}{2} \int_{\mathcal{X}} |f(x)|^2 v(dx) + \int_{\mathcal{X}} \int_{\mathbb{R}} \Big[e^{i\theta u\, f(x)} - 1 - i\theta h(u\, f(x))\Big]\, \nu(du\, dx).$$

f, and requires no arbitrary choice of $h \neq 0$. Another simple choice is the "fully compensated" expression with $h(x) = x$,

$$M[f] \equiv \int_{\mathcal{X}} f(x)\, m(dx) + \int_{\mathcal{X}} f(x)\, W(dx) + \iint_{\mathcal{X}\times\mathbb{R}} u\, f(x)\, \tilde{N}(du\, dx), \tag{51}$$

well-defined whenever Eqn. (42) holds and

$$\iint_{\mathcal{X}\times\mathbb{R}} \big(|u\, f(x)|^2 \wedge |u\, f(x)|\big)\nu(du\, dx) < \infty. \tag{52}$$

Conditions under which these Poisson integrals are well-defined, and some of their properties, are collected in the Appendix. For more details about these variations on the Lévy-Khinchine formula (Khinchine and Lévy, 1936), see Maruyama (1970), Jacod and Shiryaev (1987, *p.* 75), Sato (1999), Kallenberg (2002, *p.* 236), or Cont and Tankov (2004).

Even when $\nu(du\, dx)$ factors in the form $\nu(du)\, \mu(dx)$ it is not, in general, possible to separate in a simple way the integrals in Eqn. (43) due to the presence of the minimum. The Musielak-Orlicz spaces Φ (resp., Φ_p) of functions $f(x)$ for which $M[f]$ of Eqn. (40) is well-defined (resp., $\mathbb{E}M[f]^p < \infty$) were found by Rajput and Rosiński (1989, Theorem 3.3) and by Kwapień and Woyczyński (1992, Theorem 8.4.1). There is no difficulty, however, in the symmetric α-stable case for $0 < \alpha < 2$, which we consider in Section 5, where $m = 0$, $v = 0$, and $\nu(du\, dx) = |u|^{-\alpha-1}\, du\, \mu(dx)$. The integrability condition Eqn. (43) becomes

$$\int_{\mathcal{X}} \int_{\mathbb{R}} \big(|u\, f(x)|^2 \wedge 1\big)\, |u|^{-\alpha-1}\, du\, \mu(dx) = \int_{\mathcal{X}} |f(x)|^{\alpha}\, \mu(dx) \int_{\mathbb{R}} \big(v^2 \wedge 1\big) |v|^{-\alpha-1}\, dv < \infty$$

or, since the second integral converges for $0 < \alpha < 2$, simply

$$\int_{\mathcal{X}} |f(x)|^{\alpha}\, \mu(dx) < \infty, \tag{53}$$

with $M[f] \in L^p$ for $0 < p < \alpha$. Since the characteristic function of $M[f]$ may be expressed as

$$\log \mathbb{E}\, \{i\theta\, M[f]\} = -c_{\alpha}\, |\theta|^{\alpha} \int_{\mathbb{R}} |f(x)|^{\alpha}\, \mu(dx)$$

for some positive constant c_{α}, $M[f]$ is seen to have the symmetric α-stable (or "SαS") distribution, with heavy tails and infinite variance (see Samorodnitsky and Taqqu (1994) for details).

We now focus on downstairs and upstairs representations of stationary and stationary-increment infinitely divisible processes and provide several examples. These processes can be used to model realistic networking situations.

4.2 Stationary processes: downstairs representation

The random measure $M(dx)$ is called *stationary* on $\mathcal{X} = \mathbb{R}$ if $M[f_t]$ has the same distribution for all translates $f_t(x) \equiv f(t - x)$; this will happen if and only if $m(dx)$, $v(dx)$, and $\nu(du\, dx)$ are translation-invariant in $x \in \mathcal{X}$, *i.e.*, if

$$m(dx) = m\, dx \qquad v(dx) = v\, dx \qquad \nu(du\, dx) = \nu(du)\, dx$$

for some constants $m \in \mathbb{R}$ and $v \in \mathbb{R}_+$ and some σ-finite positive measure $\nu(du)$ on $\mathbb{R}$ where, as usual, "dx" denotes Lebesgue measure. A stationary random measure $M(dx)$ will determine a stationary stochastic process $X'_t \equiv M[f_t]$ for any kernel function $f_t(x)$ of the form $f_t(x) = h(t-x)$ that satisfies Eqns. (42, 43) for each $t \in \mathbb{R}$:

$$X'_t = \int_{\mathbb{R}} h(t-x)\, M(dx). \tag{54}$$

It is straightforward to obtain the mean and variance, when they exist. If $h \in L^1 \cap L^2(\mathbb{R}, dx)$ and $\int_{\mathbb{R}} (u^2 \wedge |u|)\, \nu(du) < \infty$, then f_t satisfies Eqn. (48) and hence Eqn. (45) and so

$$\mathbb{E}X'_t = m \int_{\mathbb{R}} h(x)\, dx + \iint_{|u\, h(x)|>1} u\, h(x)\, \nu(du)\, dx.$$

If in addition $\sigma^2 \equiv v + \int_{\mathbb{R}} u^2\, \nu(du) < \infty$, then the variance $\mathrm{Var} X'_t = \sigma^2 \int_{\mathbb{R}} h(x)^2\, dx$ is finite and the covariance is given by

$$\rho(t-s) = \mathrm{Cov}(X'_s, X'_t) = \sigma^2 \int_{\mathbb{R}} h(s-x)\, h(t-x)\, dx.$$

In this case the spectral density is given by

$$\chi(\theta) \;=\; \iint_{\mathbb{R}^2} e^{i\theta t} \sigma^2 h(s-x)\, h(t-x)\, dx\, dt = \sigma^2 |\hat{h}(\theta)|^2,$$

proportional to the squared modulus of $\hat{h}(\theta)$. Simple illustrative examples of stationary ID processes and ID processes with stationary increments, all with $m = 0$, include:

$$\begin{array}{rlll}
\textbf{OU}: & v = \sigma^2 & \nu(du) = 0 & f_t(x) = \sqrt{2\lambda}\sigma\, e^{-\lambda(t-x)} 1_{\{x \le t\}} \\
\textbf{fOU}: & v = \sigma^2 & \nu(du) = 0 & f_t(x) = \sqrt{2}\lambda^{\kappa - \frac{1}{2}} (t-x)_+^{\kappa-1} e^{-\lambda(t-x)} / \Gamma(\kappa) \\
\textbf{OUL}: & v = 0 & \nu(du) > 0 & f_t(x) = \sqrt{2\lambda}\sigma\, e^{-\lambda(t-x)} 1_{\{x \le t\}} \\
\textbf{fOUL}: & v = 0 & \nu(du) > 0 & f_t(x) = \sqrt{2}\lambda^{\kappa - \frac{1}{2}} (t-x)_+^{\kappa-1} e^{-\lambda(t-x)} / \Gamma(\kappa).
\end{array}$$

4.2.1 Example: OUL generated by a gamma random measure

A gamma-distributed random variable $Y \sim \mathsf{Ga}(\alpha, \beta)$ with shape parameter $\alpha > 0$ and rate parameter $\beta > 0$ (hence mean α/β) is a non-negative random variable with density function given by

$$y^{\alpha-1} e^{-\beta y} \beta^\alpha / \Gamma(\alpha), \qquad y > 0.$$

Its log characteristic function is

$$\log \mathbb{E} e^{i\theta Y} \;=\; \log(1 - i\theta/\beta)^{-\alpha} = \int_{\mathbb{R}_+} \left(e^{i\theta u} - 1\right) \nu(du)$$

with Lévy measure

$$\nu(du) \;=\; \alpha e^{-\beta u} 1_{\{u>0\}} u^{-1}\, du \tag{55}$$

(see Rogers and Williams, 1994, I28). We shall now construct an explicit example of the downstairs OUL processes. Consider the *gamma random measure* $M(dx)$ with Lévy measure given by Eqn. (55), that is, $M(dx)$ in Eqn. (39) assigns independent gamma-distributed random variables

$M(A_j) \sim \mathsf{Ga}(\alpha|A_j|, \beta)$ to disjoint sets $A_j \in \mathbb{R}$ of finite Lebesgue measure $|A_j|$. The log characteristic function of the resulting process

$$X'_t \equiv \sqrt{2\lambda} \int_{-\infty}^{t} e^{-\lambda(t-x)}\, M(dx)$$

is given by Eqn. (32) as

$$\begin{aligned}
\log \mathbb{E}\big[e^{i\theta X'_t}\big] &= \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \Big[\exp\Big(i\,\theta u\, e^{-x}\sqrt{2\lambda}\Big) - 1\Big]\; \alpha e^{-\beta u} u^{-1}\, du\, dx/\lambda \\
&= \int_{\mathbb{R}_+} \big[e^{i\theta w} - 1\big]\, \frac{\alpha}{\lambda}\, E_1\big(w\,\beta/\sqrt{2\lambda}\big)\, dw/w,
\end{aligned}$$

where $E_1(z) \equiv \int_z^\infty e^{-t}\, dt/t$ is the Exponential Integral function (Abramowitz and Stegun, 1964, Sec. 5.1). Thus X'_t has an ID distribution with Lévy measure

$$\tilde{\nu}(dw) = \frac{\alpha}{\lambda} E_1\big(w\,\beta/\sqrt{2\lambda}\big) 1_{\{w>0\}}\, dw/w.$$

Note that although the distribution of X'_t is ID, it is *not* from the gamma family because its Lévy measure is not of the form given in Eqn. (55). The covariance of this stationary process is given by Eqn. (5), $\rho(s,t) = \sigma^2\, e^{-\lambda|t-s|}$, with $\sigma^2 = \int_{\mathbb{R}} u^2 \nu(du) = \alpha/\beta^2$.

4.2.2 Example: fOUL generated by a SαS random measure

Now consider the fOUL process given in Eqn. (27), here rescaled in the form

$$X'_t = \Gamma(\kappa)\, Y_t^\kappa = \int_{\mathbb{R}} f_t(x) M(dx) \quad \text{with} \quad f_t(x) = \sqrt{2}\, \lambda^{\kappa - \frac{1}{2}} (t-x)_+^{\kappa-1}\, e^{-\lambda(t-x)}$$

where the SαS measure $M(dx)$ has Lévy measure $\nu(du\, dx) = |u|^{-\alpha-1}\, du\, dx$ for some $\alpha \in (0,2)$. This is well-defined by Eqn. (53) provided

$$\begin{aligned}
\int_{\mathbb{R}} \Big|f_t(x)\Big|^\alpha dx &= \Big[\sqrt{2}\,\lambda^{\kappa-\frac{1}{2}}\Big]^\alpha \int_{-\infty}^{t} (t-x)^{\alpha(\kappa-1)} e^{-\alpha\lambda(t-x)}\, dx \\
&= \Big[\sqrt{2}\,\lambda^{\kappa-\frac{1}{2}}\Big]^\alpha \int_0^\infty r^{\alpha(\kappa-1)} e^{-\alpha\lambda r}\, dr \\
&= \Big[\sqrt{2\lambda}\,\alpha^{1-\kappa}\Big]^\alpha\, \Gamma\big(1 + \alpha(\kappa-1)\big) \qquad (56) \\
&< \infty,
\end{aligned}$$

i.e., for $\kappa > 1 - 1/\alpha$ (the integral diverges for $\kappa \le 1 - 1/\alpha$). By Eqns. (29–31), X'_t is ID with Lévy measure

$$\tilde{\nu}(du) \;=\; \int \big|u/f_t(x)\big|^{-\alpha}\, dx\, du/u \;=\; \Big[\sqrt{2\lambda}\,\alpha^{1-\kappa}\Big]^\alpha\, \Gamma\big(1+\alpha(\kappa-1)\big)\, |u|^{-1-\alpha}\, du,$$

again SαS but with a different scale constant.

4.3 Processes with stationary increments: downstairs representation

The indefinite integral $X_t = \int_0^t X'_s \, ds$ of Eqn. (54) gives a process X_t with stationary increments that may be expressed as $M[f_t]$ for a kernel of the form $f_t(x) = \int_0^t h(s-x)\, ds = [H(t-x) - H(-x)]$:

$$X_t \equiv M[f_t] = \int_{\mathbb{R}} \big[H(t-x) - H(-x)\big]\, M(dx). \tag{57}$$

The conditions imposed on $f_t(x) = [H(t-x) - H(-x)]$ by Eqn. (43) are that, *if* $m \neq 0$, then $f_t(x) \in L^1(\mathbb{R}, dx)$, and that *if* $v > 0$ or $\nu > 0$ for $\mathrm{Var} X_t < \infty$, then $f_t(x) \in L^2(\mathbb{R}, dx)$ as well. This includes some kernels $f_t(x) = [H(t-x) - H(-x)]$ for which $H(x)$ is not of the form $\int_0^t h(s-x)\, ds$ for $h \in L^1(\mathbb{R})$ or $h \in L^2(\mathbb{R})$ as above, and hence some processes X_t with independent increments which do not have differentiable sample paths. For example, for any $\kappa \in (-\frac{1}{2}, \frac{1}{2})$ the kernel $f_t(x) \equiv \big[(t-x)_+^\kappa - (-x)_+^\kappa\big]$ of fBM satisfies $f_t \in L^2(\mathbb{R}, dx)$, while no power of the derivative $h(r) = H'(r) = \kappa (r)_+^{\kappa-1}$ of $H(r) = (r)_+^\kappa$ is integrable over $(\mathbb{R}, dx)$. Thus, examples of ID processes with stationary increments, both with $m = 0$, include:

$$\begin{aligned}
&\textbf{BM}: \quad v = 1 \qquad \nu(du) = 0 \quad f_t(x) = 1_{(0,t]}(x)\\
&\textbf{fBM}: \quad v = \sigma^2 \quad \nu(du) = 0 \quad f_t(x) = [(t-x)_+^\kappa - (-x)_+^\kappa].
\end{aligned}$$

4.4 Stationary processes: upstairs representation

Now let $M(dx)$ be a random measure on $\mathcal{X} = \mathbb{R}^2$ whose distribution is invariant under translation[4] of $x = (x_1, x_2) \in \mathcal{X}$. This will happen if and only if $m(dx)$, $v(dx)$, and $\nu(du\, dx)$ are translation-invariant in x, *i.e.*, if each is a product measure of two-dimensional Lebesgue measure $dx = dx_1\, dx_2$ and a constant or σ-finite measure in the other variable(s),

$$m(dx) = m\, dx_1\, dx_2 \quad v(dx) = v\, dx_1\, dx_2 \quad \nu(du\, dx) = \nu(du)\, dx_1\, dx_2. \tag{58}$$

Such a random measure will determine a stationary stochastic process $X'_t \equiv M[f_t]$ for any kernel function $f_t(x)$ of the form $f_t(x) = 1_{\{0 < x_1 \le g(t-x_2)\}}$ for suitable non-negative measurable functions $g : \mathbb{R} \to \mathbb{R}_+$,

$$X'_t = \int_{\mathbb{R}} \int_0^{g(t-x_2)} M(dx), \tag{59}$$

provided $\int (u^2 \wedge 1)\, \nu(du) < \infty$. The process is well-defined (*i.e.*, satisfies Eqns. (42, 43)) for any non-negative measurable $g \in L^1(\mathbb{R})$, for then $f_t = 1_{\{0 < x_1 \le g(t-x_2)\}} \in L^1 \cap L^2(\mathcal{X}, dx)$ because

$$\begin{aligned}
\int_{\mathcal{X}} |f_t(x)|^p\, dx &= \int_{\mathcal{X}} 1_{\{0 < x_1 \le g(t-x_2)\}} dx_1\, dx_2 \qquad \text{for any } p > 0\\
&= \int_{\mathbb{R}} g(t-x_2)\, dx_2\\
&= \int_0^\infty g(x)\, dx < \infty.
\end{aligned}$$

[4] Only translation-invariance in the second component x_2 of x is required for the construction below, but we impose translation-invariance in both components to simplify the exposition.

The mean, variance, and covariance of X'_t, where defined, are given by

$$\begin{aligned}
\mathbb{E}X'_t &= m\int_0^\infty g(x)\,dx + \iint_{|u\,f_t(x)|>1} u\,f_t(x)\,\nu(du)\,dx \\
\mathrm{Var}X'_t &= \sigma^2\int_0^\infty g(x)\,dx \\
\mathrm{Cov}(X'_s, X'_t) &= \sigma^2\int_{|t-s|}^\infty g(x)\,dx \qquad (60)
\end{aligned}$$

where $\sigma^2 = v + \int_{\mathbb{R}} u^2\,\nu(du)$; the mean is well-defined provided $g \in L^1(\mathbb{R}_+)$ and $\int_{\mathbb{R}} |u|\nu(du) < \infty$, the variance and covariance provided $g \in L^1(\mathbb{R}_+)$ and $\int_{\mathbb{R}}(u^2 \wedge |u|)\nu(du) < \infty$ (see Eqn. (48)).

Eqn. (60) implies that $g(t)$ and the covariance function $\rho(t) = \mathrm{Cov}(X'_s, X'_{t+s})$ are related by

$$g(t) = -\sigma^{-2}\,\frac{d}{dt}\rho(t), \quad t > 0 \qquad (61)$$

as in Eqn. (38) and, in the Gaussian case, Eqn. (21).

4.4.1 Modeling realistic covariances

Covariance functions $\rho(t-s) = \mathrm{Cov}(X'_s, X'_t)$ estimated from observations of packet arrival rates in data communications networks frequently exhibit an initial exponential fall-off $\rho(t) \approx \sigma^2 e^{-\lambda|t|}$ for small $|t|$, followed by a more gradual fall-off $\rho(t) \sim c|t|^{-\alpha-1}$ as $t \to \pm\infty$. Since Eqn. (60) is valid for any integrable decreasing function $g(t)$, and since $g(t)$ and $\rho(t)$ are related by Eqn. (61), this behavior can be modeled by an upstairs representation in which $g(t)$ falls off exponentially at first, then like a power.

For an example, consider $\rho(t) \propto (1+\lambda|t|/\alpha)^{-\alpha}$ for $\alpha > 0$ (arising from the upstairs construction with $g(t) \propto (1+\lambda t/\alpha)^{-\alpha-1}$) or, more flexibly,

$$\begin{aligned}
-\big[\log g(t)\big]' &= \frac{\lambda\delta^2 + (\alpha+1)t}{\delta^2 + t^2}, \quad t > 0 \qquad (62) \\
g(t) &= \lambda(1 + t^2/\delta^2)^{-(\alpha+1)/2} e^{-\lambda\delta\arctan(t/\delta)} \qquad (63)
\end{aligned}$$

From Eqn. (62) it is apparent that the derivative of $\log g(t)$ is approximately a constant λ for $t \ll \delta$ and is approximately $(\alpha+1)/t$ for $t \gg \delta$, so we may expect that $g(t) \approx \lambda\exp(-\lambda t)$ for small t and $g(t) \approx ct^{-\alpha-1}$ for large t, easily verified from Eqn. (63). This leads to the behavior described above ($\rho \approx e^{-\lambda|t|}$ for $t \ll \delta$, $\rho \approx c|t|^{-\alpha}$ for $t \gg \delta$) of $\rho(t) = \sigma^2\int_{|t|}^\infty g(x)\,dx$, whose logarithm is plotted in Fig. (2) along with a dashed line indicating the logarithm of $\exp(-\lambda|t|)$ for comparison. A stationary process with this covariance is easily constructed via the upstairs representation of Eqn. (59) using $g(t)$ from Eqn. (63).

4.4.2 Example: Upstairs OUL generated by a gamma random measure

For an explicit example of the upstairs OUL processes, with exponential decay in the covariance $\rho(t-s)$, consider (as in Section 4.2.1) the gamma random measure $M(dx_1\,dx_2)$, with Lévy measure

$$\nu(du) = \alpha e^{-\beta u} 1_{\{u>0\}}\,du/u,$$

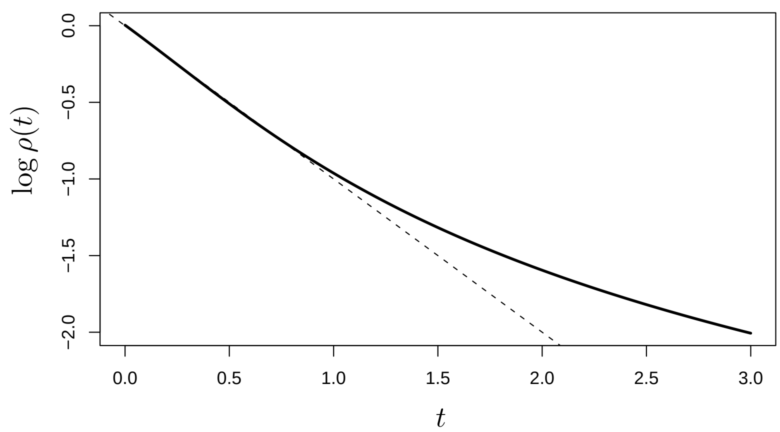

Figure 2: Realistic log covariance function $\log \rho(t)$ (solid), and $\log \big[\exp(-\lambda|t|)\big]$ (dashed) for contrast. The covariance behaves like an exponential for small t, like a power for large t.

so that $M(dx)$ in Eqn. (59) assigns independent gamma-distributed random variables $M(A_j) \sim \mathsf{Ga}(\alpha|A_j|, \beta)$ to disjoint sets $A_j \in \mathbb{R}^2$ of Lebesgue measure $|A_j|$. The log characteristic function of the upstairs process with $g(r) = \lambda e^{-\lambda r} 1_{\{r>0\}}$,

$$X'_t \equiv \int_{-\infty}^{t} \int_{0}^{\lambda\, e^{-\lambda(t-x_2)}} M(dx_1\, dx_2)$$

is given by Eqn. (36) as

$$\log \mathbb{E}\big[e^{i\theta X_t}\big] = \int_{\mathbb{R}_+} \big(e^{i\theta u} - 1\big)\, \alpha\, e^{-\beta\, u}\, du/u.$$

Thus X'_t has a gamma $\mathsf{Ga}(\alpha, \beta)$ distribution, in contrast to the downstairs-representation presented in Section 4.2.1, even though both (by Eqn. (60)) have the same covariance function

$$\rho(s,t) = \int_{\mathbb{R}} u^2\, \nu(du) \int_{|t-s|}^{\infty} \lambda e^{-\lambda x} dx = \frac{\alpha}{\beta^2} e^{-\lambda|t-s|}.$$

4.5 Processes with stationary increments: upstairs representation

The indefinite integral $X_t = \int_0^t X'_s\, ds$ of Eqn. (59) gives a process X_t with stationary increments which may be expressed as $X_t \equiv M[f_t]$ for a kernel of the form

$$f_t(x_1, x_2) \quad = \quad \int_0^t 1_{\{0<x_1 \le g(s-x_2)\}}\, ds \quad = \quad \big|(0,t] \cap \{s:\ g(s-x_2) \ge x_1\}\big|, \tag{64}$$

thus

$$X_t \quad = \quad \int_{\mathbb{R}^2} \big|(0,t] \cap \{s:\ g(s-x_2) \ge x_1\}\big|\, M(dx_1\, dx_2) \tag{65}$$

But it is not necessary that the derivative X_t' exist for X_t to exist; the function g in Eqn. (65) has only to satisfy the conditions implied by Eqn. (43). Conditions for X_t to have finite variance are more restrictive; we state sufficient conditions in the following proposition.

Proposition 4.1 *For any $v \in \mathbb{R}_+$, any nonnegative measure ν on $\mathbb{R}$ satisfying $\int_{\mathbb{R}} u^2 \nu(du) < \infty$, and any function $g : \mathbb{R} \to \mathbb{R}_+$ that vanishes on $(-\infty, 0)$, decreases on $[0, \infty)$, and satisfies*

$$\int_0^\infty (x \wedge 1) g(x)\, dx < \infty \tag{66}$$

we can construct an ID random measure $M(dx)$ on $\mathcal{X} = \mathbb{R}^2$ of Eqn. (39) and a finite-variance process X_t with stationary increments

$$\begin{aligned} X_t &= M[f_t] \\ &= \int_{\mathcal{X}} \big|(0, t] \cap \{s : \ g(s - x_2) \geq x_1\}\big|\, M(dx) \\ &= \int_{\mathcal{X}} f_t(x)\, W(dx) + \int_{\mathcal{X}} \int_{\mathbb{R}} u\, f_t(x)\, \tilde{N}(du\, dx). \end{aligned} \tag{67}$$

where $f_t(x) = \big|(0, t] \cap \{s : \ g(s - x_2) \geq x_1\}\big|$. Moreover, in this case, $\mathbb{E}X_t = 0$ and the variance and covariance of X_t are given by

$$\begin{aligned} \mathrm{Var} X_t &= \sigma^2 \int_0^\infty [t^2 - (t - x)_+^2]\, g(x)\, dx \\ \mathrm{Cov}(X_s, X_t) &= \frac{1}{2}\big[\mathrm{Var} X_t + \mathrm{Var} X_s - \mathrm{Var} X_{|t-s|}\big] \\ &= \frac{\sigma^2}{2} \int_0^\infty \big[2st + (|s - t| - x)_+^2 - (s - x)_+^2 - (t - x)_+^2\big]\, g(x)\, dx \end{aligned}$$

where $\sigma^2 \equiv v + \int_{\mathbb{R}} u^2\, \nu(du)$.

Proof. The Gaussian integral in Eqn. (67) is well-defined for any $f_t \in L^2(\mathcal{X}, dx)$. So is the (fully compensated) Poisson integral in Eqn. (67), because Eqn. (52) holds whenever $\int_{\mathbb{R}} u^2 \nu(du) < \infty$ and $\int_{\mathcal{X}} |f(x)|^2\, dx < \infty$. Now the condition $f_t \in L^2(\mathcal{X}, dx)$ becomes

$$\begin{aligned} \int_{\mathcal{X}} f_t(x)^2\, dx &= \int_{\mathcal{X}} \left[\int_0^t \int_0^t 1_{\{0 < x_1 \leq g(s_1 - x_2)\}} 1_{\{0 < x_1 \leq g(s_2 - x_2)\}}\, ds_1\, ds_2\right] dx \\ &= \iint_{\mathbb{R}^2} \left[2 \iint_{(x_2 \vee 0) < s_1 < s_2 < t} 1_{\{0 < x_1 \leq g(s_2 - x_2)\}}\, ds_1\, ds_2\right] dx_1\, dx_2 \\ &= \int_{-\infty}^t \left[2 \int_0^t [s_2 - (x_2 \vee 0)]\, g(s_2 - x_2)\, ds_2\right] dx_2 \\ &= \int_{-\infty}^0 \int_0^t 2\, s\, g(s - z)\, ds\, dz + \int_0^t \int_z^t 2(s - z)\, g(s - z)\, ds\, dz \\ &= \int_0^\infty \int_{-x}^{(t-x) \wedge 0} 2\, (x + z)\, g(x)\, dz\, dx + \int_0^t \int_0^{t-x} 2x\, g(x)\, dz\, dx \\ &= \int_0^\infty (x^2 \wedge t^2)\, g(x)\, dx + \int_0^t 2(t - x)\, x\, g(x)\, dx \\ &= \int_0^\infty [t^2 - (t - x)_+^2]\, g(x)\, dx < \infty, \end{aligned} \tag{68}$$

which is satisfied for example if Eqn. (66) holds. Eqn. (47) yields the given expression $\mathrm{Var}X_t$, while that for the covariance $\mathrm{Cov}(X_s, X_t)$ follows from the polarization identity. ∎

4.6 Poissonized Telecom and fBM Processes

Condition Eqn. (66) is less restrictive than requiring that $g \in L^1$ — for example, it is satisfied for any power

$$g(x) = x_+^{2H-3} \tag{69}$$

in the range $\frac{1}{2} < H < 1$ and in this case Eqn. (68) gives $\sigma_t^2 = c\,t^{2H}$, for $c = \sigma^2/2H(2H-1)(1-H)$. The resulting process

$$X_t = \iint_{\mathbb{R}^2} \left[\int_0^t 1_{\{0<x_1\le(s-x_2)_+^{2H-3}\}} ds \right] M(dx_1\,dx_2) \tag{70}$$

is called the Poissonized Telecom process (Cohen and Taqqu, 2004).

In the Gaussian case ($v = 1$, $\nu(du) = 0$, $M(dx) = W(dx)$), this process

$$X_t = \iint_{\mathbb{R}^2} \left[\int_0^t 1_{\{0<x_1\le(s-x_2)_+^{2H-3}\}} ds \right] W(dx_1\,dx_2) \tag{71}$$

gives the fBM process.

Corollary 5.2 below shows that a similar construction leads to the infinite-variance Telecom process when $M(dx)$ is a symmetric α-stable random measure ($0 < \alpha < 2$), but the exponent will be $\alpha H - \alpha - 1$ rather than $2H - 3$.

4.7 The random field version

Fractional Brownian motion and the Poissonized Telecom process are two examples of processes that have stationary increments but are not differentiable and so cannot be written as integrals of well-defined stationary processes. As indicated in Section 2.1 in the case of fractional Brownian motion, it is possible to alleviate this problem by working with random fields. We now build random field versions of the ID processes X_t defined in Eqns. (65, 67).

For suitable functions $\phi(t)$ we can construct a *random field version* $X[\phi]$ of the ID process X_t defined in Section 4.5 by setting

$$\begin{aligned} X[\phi] &\equiv \int_{\mathcal{X}} \int_{\mathbb{R}} \phi(s) 1_{\{0<x_1\le g(s-x_2)\}}\, ds\, M(dx) && (72) \\ &= \int_{\mathcal{X}} f_\phi(x)\, W(dx) + \int_{\mathcal{X}} \int_{\mathbb{R}} u\, f_\phi(x)\, \tilde{N}(du\, dx) && (73) \end{aligned}$$

where $f_\phi(x) \equiv \int_{\mathbb{R}} \phi(s) 1_{\{0<x_1\le g(s-x_2)\}}\, ds$ and where we suppose $v(dx) = v\, dx$ and $\int_{\mathbb{R}} u^2 \nu(du) < \infty$ as in Proposition 4.1. The condition for $X[\phi]$ to be well-defined follows from Eqn. (48), namely,

$f_\phi(x) \in L^2(\mathcal{X}, dx)$, leading to the requirement that

$$\begin{aligned}
\int_{\mathcal{X}} f_\phi(x)^2\,dx &= \int_{\mathcal{X}} \left[\iint_{\mathbb{R}^2} \phi(s_1)\phi(s_2) 1_{\{0<x_1\le g(s_1-x_2)\}} 1_{\{0<x_1\le g(s_2-x_2)\}}\,ds_1\,ds_2\right] dx \\
&= 2 \iiiint_{x_2<s_1<s_2<\infty} \phi(s_1)\phi(s_2) 1_{\{0<x_1\le g(s_2-x_2)\}}\,ds_1\,ds_2\,dx_1\,dx_2 \\
&= \iint_{\mathbb{R}_+} \phi(s_1)\,\phi(s_2) \left[\int_{|s_1-s_2|}^\infty g(r)\,dr\right] ds_1\,ds_2 \\
&= \int_0^\infty \left[\iint_{|s_1-s_2|<r} \phi(s_1)\,\phi(s_2)\,ds_1\,ds_2\right] g(r)\,dr < \infty.
\end{aligned}$$

Since the bracketed term is bounded by both $2r\|\phi\|_\infty\|\phi\|_1$ and by $\|\phi\|_2^2$, we have

Proposition 4.2 *$X[\phi]$ is well-defined for all uniformly bounded integrable functions $\phi \in L^1(\mathbb{R})$ and all non-increasing non-negative functions $g : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\int_{\mathbb{R}} (r \wedge 1) g(r)\,dr < \infty$.*

For $H \in (\frac{1}{2}, 1)$ and $g(r) = r_+^{2H-3}$, this yields the random-field version of the fBM process, for $v = \sigma^2$ and $\nu(du) = 0$. Since the functions $\phi(s) = \sum \theta_j 1_{(0,t_j]} \in L^1(\mathbb{R})$ are integrable, this shows the processes we generated in Eqn. (65) are all well-defined.

It follows that

$$G(\theta) \equiv \int_{\mathbb{R}} e^{i\theta r} \left[\int_{|r|}^\infty g(u)\,du\right] dr = \frac{2}{\theta} \int_0^\infty \sin(\theta r) g(r)\,dr$$

is finite for all $\theta \neq 0$; the largest collection Φ of functions $\phi \in L^1$ for which $X[\phi]$ is well-defined will be those for which

$$\begin{aligned}
\int_{\mathcal{X}} f_\phi(x)^2\,dx &= \iint_{\mathbb{R}_+} \phi(s_1)\,\phi(s_2) \left[\int_{|s_1-s_2|}^\infty g(r)\,dr\right] ds_1\,ds_2 \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{\phi}(\theta)|^2\, G(\theta)\,d\theta < \infty.
\end{aligned}$$

For example, for $g(r) = r_+^{2H-3}$ as in Eqn. (69) we have $G(\theta) = |\theta|^{1-2H}\Gamma(2H-1)\sin(\pi H)/(1-H)$, and we can construct $X[\phi]$ for any ϕ satisfying

$$\int_{\mathbb{R}} |\hat{\phi}(\theta)|^2\, |\theta|^{1-2H}\,d\theta < \infty.$$

This includes all $\phi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ since $\hat{\phi}$ is then uniformly bounded (controlling the integral over $|\theta| \le 1$) and in $L^2(\mathbb{R})$ (controlling it over $|\theta| \ge 1$). One has $\mathbb{E}X_\phi = 0$ and

$$\begin{aligned}
\mathrm{Var} X_\phi &= \sigma^2 \int_0^\infty \left[\iint_{|y_1-y_2|<r} \phi(y_1)\phi(y_2)\,dy_1\,dy_2\right] g(r)\,dr, \\
\mathrm{Cov}(X_\phi, X_\psi) &= \sigma^2 \int_0^\infty \left[\iint_{|y_1-y_2|<r} \phi(y_1)\psi(y_2)\,dy_1\,dy_2\right] g(r)\,dr,
\end{aligned}$$

where again $\sigma^2 \equiv v + \int_{\mathbb{R}} u^2\,\nu(du)$, provided the corresponding integrals converge.

5 The Telecom process

Renewal-reward processes with heavy tailed inter-renewal times were introduced by Mandelbrot (1969) in an economics context. Mandelbrot's model was developed by Taqqu and Levy (1986) and it was adapted to the telecommunication context by Taqqu *et al.* (1997) who considered an "on-off" version of the model. If the number of replications grows to infinity then, after subtracting the mean and rescaling, the limit turns out to be fractional Brownian motion with self-similarity (Hurst) exponent

$$H = \frac{3-\gamma}{2}, \tag{74}$$

where $1 < \gamma < 2$ is the heavy-tail exponent of the inter-renewal times.

Levy and Taqqu (2000) studied what happens when the rewards are heavy-tailed as well, with exponent $\alpha \in (\gamma, 2)$ (they are in the domain of attraction of a stable distribution with index γ). They showed that, after centering and rescaling, the limit process $\{Z_t, t \geq 0\}$, called the *Telecom process*, is a symmetric α-stable (SαS) process with stationary increments which is self-similar with exponent

$$H = \frac{\alpha - \gamma + 1}{\alpha}. \tag{75}$$

Further results about the Telecom process can be found in Kaj and Taqqu (2004).

5.1 Downstairs and upstairs representations

It was shown by Pipiras and Taqqu (2000) that the Telecom process can be represented in the form:

$$Z_t = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\int_0^t 1_{\{v_1 < s \leq v_2\}} ds \right] M(dv_1\, dv_2), \tag{76}$$

where M is a SαS random measure on $\mathbb{R}^2$ with a control measure

$$m(dv_1\, dv_2) = \gamma (v_2 - v_1)_+^{-\gamma-1} dv_1 dv_2, \tag{77}$$

that is, for any set A in $\mathbb{R}^2$, the characteristic function of the SαS random variable $M(A)$ is

$$\mathbb{E} \exp\{i\theta M(A)\} = \exp\{-|\theta|^\alpha m(A)\}.$$

The measure m plays the role of scale factor (it weighs different sets A in different ways). For more details about SαS random measures and their control measures, see Samorodnitsky and Taqqu (1994, Chapter 3).

Observe that the Telecom self-similarity exponent H of Eqn. (75) would agree with that for fBM given (for $\alpha = 2$) in Eqn. (74)— and, in fact, the Telcom process converges to the Gaussian fBM as $\alpha \nearrow 2$.

The Telecom process has a corresponding integral representation with respect to a compensated Poisson measure $\tilde{N}$:

$$Z_t \stackrel{d}{=} \iiint_{\mathbb{R}^3} \left[\int_0^t 1_{\{v_1 < s \leq v_2\}} ds \right] u\, \tilde{N}(du\, dv_1\, dv_2) \tag{78}$$

where $\tilde{N}(du\, dv_1\, dv_2) = N(du\, dv_1\, dv_2) - \nu(du\, dv_1\, dv_2)$ and $N(du\, dv_1\, dv_2) \sim \mathsf{Po}\big(\nu(du\, dv_1\, dv_2)\big)$, with

$$\nu(du\, dv_1\, dv_2) = |u|^{-\alpha-1} du\, m(dv_1\, dv_2), \tag{79}$$

where $m(dv_1\, dv_2) = \gamma(v_2 - v_1)_+^{-\gamma-1} dv_1 dv_2$ is the Telecom control measure of Eqn. (77).

The representations of Eqns. (76, 78) are not easy to interpret, particularly the form of the control measure m in Eqn. (77). Our goal is to obtain an upstairs representation of the Telecom process which has a direct networking interpretation. Since the Telecom becomes fBM when $\alpha = 2$, this will also provide a physical explanation for why fBM appears naturally in the context of network traffic (where fBM appears commonly).

We will use the following result to obtain the upstairs representation:

Proposition 5.1 *Let $g(s)$, $s \geq 0$ be a non-negative differentiable strictly monotone decreasing function on $\mathbb{R}_+$ with $g(0) < \infty$, and set $g(s) = 0$ for $s < 0$. Then the process*

$$X_t^{(1)} = \iint_{\mathbb{R}^2} \left[\int_0^t 1_{\{v_1 < r \leq v_2\}} dr \right] M_\alpha^{(1)}(dv_1\, dv_2), \tag{80}$$

where $M_\alpha^{(1)}$ is a $S\alpha S$ random measure on $\mathbb{R}^2$ with control measure

$$m^{(1)}(dv_1\, dv_2) = |g'(v_2 - v_1)|\, dv_1\, dv_2,$$

has the same finite-dimensional distributions as the process

$$X_t^{(2)} = \int_{\mathbb{R}} \int_0^\infty \left[\int_0^t 1_{\{0 < x_1 \leq g(s - x_2)\}} ds \right] M_\alpha^{(2)}(dx_1\, dx_2), \tag{81}$$

where $M_\alpha^{(2)}$ is a $S\alpha S$ random measure on $\mathbb{R}^2$ with control measure

$$m^{(2)}(dx_1\, dx_2) = dx_1\, dx_2,$$

provided either process is well-defined. If the process $X_t^{(2)}$ is differentiable, that is, if

$$\begin{aligned} X_t' &= \int_0^\infty \int_{\mathbb{R}} 1_{\{0 < x_1 \leq g(t - x_2)\}} M_\alpha^{(2)}(dx_1\, dx_2) \\ &= \int_{-\infty}^t \int_0^{g(t - x_2)} M_\alpha^{(2)}(dx_1\, dx_2) \end{aligned} \tag{82}$$

is well-defined, then $X^{(1)}$ and $X^{(2)}$ have the same finite-dimensional distributions as

$$\begin{aligned} X_t^{(3)} &= \int_0^t X_s'(ds) \\ &= \int_0^t \int_{\mathbb{R}} \int_0^{g(s - x_2)} M_\alpha^{(2)}(dx_1\, dx_2)\, ds \qquad (83) \\ &\stackrel{d}{=} \int_0^t \int_{\mathbb{R}} \int_0^{g(s - x_2)} \int_{\mathbb{R}} u\, \tilde{N}(du\, dx_1\, dx_2)\, ds, \qquad (84) \end{aligned}$$

where $\tilde{N}(du\, dx_1\, dx_2) = N(du\, dx_1\, dx_2) - \nu(du\, dx_1\, dx_2)$ is the compensated version of a Poisson random measure $N(du\, dx_1\, dx_2) \sim \mathsf{Po}(\nu(du\, dx_1\, dx_2))$ with control (or mean) measure

$$\nu(du\, dx_1\, dx_2) = |u|^{-\alpha-1} du\, dx_1\, dx_2. \tag{85}$$

Proof. Let $\theta_1, ..., \theta_n$ be arbitrary real numbers and $t_1, ..., t_n$ arbitrary positive numbers. The finite dimensional distributions of $X^{(1)}$ are characterized by

$$-\log \mathbb{E} \exp \left\{ i \sum_{j=1}^{n} \theta_j X_{t_j}^{(1)} \right\} = A^{\alpha}$$

where

$$A^{\alpha} \equiv \iint_{\mathbb{R}^2} \left| \sum_{j=1}^{n} \theta_j \int_0^{t_j} 1_{\{v_1 < r \le v_2\}} dr \right|^{\alpha} |g'(v_2 - v_1)| \, dv_1 \, dv_2$$

and those of $X^{(2)}$ by

$$-\log \mathbb{E} \exp \left\{ i \sum_{j=1}^{n} \theta_j X_{t_j}^{(2)} \right\} = B^{\alpha}$$

where

$$B^{\alpha} \equiv \int_0^{\infty} \int_{\mathbb{R}} \left| \sum_{j=1}^{n} \theta_j \int_0^{t_j} 1_{\{0 < x_1 \le g(s - x_2)\}} ds \right|^{\alpha} dx_1 \, dx_2.$$

The processes $X^{(1)}$ and $X^{(2)}$ are well-defined because $1 < \alpha < \gamma \le 2$ implies $A^{\alpha} < \infty$ and $B^{\alpha} < \infty$. To conclude $X^{(1)} \stackrel{d}{=} X^{(2)}$ we need only verify that $A^{\alpha} = B^{\alpha}$.

Observe first that by changing variables to $x_2 = v_1$ and $\tau = v_2 - v_1$, we find

$$A^{\alpha} = \iint_{\mathbb{R}^2} \left| \sum_{j=1}^{n} \theta_j \int_0^{t_j} 1_{\{x_2 < r \le x_2 + \tau\}} dr \right|^{\alpha} |g'(\tau)| \, dx_2 \, d\tau.$$

Now set

$$x_1 = g(\tau), \tag{86}$$

and observe that $1_{\{x_2 < s \le x_2 + \tau\}} = 1$ if and only if $x_2 < s \le x_2 + \tau$, if and only if $0 < s - x_2 \le \tau$, if and only if $g(0) > g(s - x_2) \ge x_1 = g(\tau) > 0$ (since $g(x)$ is strictly monotone decreasing on $\mathbb{R}_+$), if and only if $1_{\{0 < x_1 \le g(s - x_2)\}} = 1$. Since the differentials for this change of variables are related by $dx_1 = |g'(\tau)| \, d\tau$, we have $A^{\alpha} = B^{\alpha}$.

If X'_t in Eqn. (82) is well-defined, that is, if

$$\int_{\mathbb{R}} \int_0^{g(r - x_2)} dx_1 \, dx_2 = \int_0^{\infty} g(s) \, ds < \infty,$$

then one can interchange integration (Samorodnitsky and Taqqu, 1994, Theorem 11-4.1) and write $X_t = \int_0^t X'_r \, dr$. Then $X_t^{(2)}$ can be expressed a.s. as in Eqn. (83). To get Eqn. (84), express (as usual) the SαS $M_{\alpha}^{(2)}(dx_1 \, dx_2)$ in terms of a compensated Poisson measure $M_{\alpha}^{(2)}(dx_1 \, dx_2) = \int_{\mathbb{R}} u \, \tilde{N}(du \, dx_1 \, dx_2)$ with control measure as in Eqn. (85), where u corresponds to the size of the "jumps". ∎

5.2 Networking interpretation

The Poisson measure N in Eqn. (84) generates points (u, x_1, x_2) with the following network interpretation. Each point (u, x_1, x_2) indicates the arrival at some network component at time x_2 of a job of intensity (rate) u, which lasts for a duration τ with survival function (complimentary CDF) $g(t)/g(0)$, *i.e.*,

$$\mathbb{P}[\tau > t] = g(t)/g(0).$$

Thus $x_1 = g(\tau)$ may be regarded as the realization of a uniform random variable on the interval $(0, g(0)]$. Hence

> *The event $\{0 < x_1 \le g(r - x_2)\}$ is the event that the job generated at time x_2 has not yet terminated at time r.*

Given r and the starting time x_2, this event occurs with conditional probability

$$\mathbb{P}[0 < x_1 \le g(r - x_2) \mid x_2] = \frac{g(r - x_2)}{g(0)},$$

which vanishes for $r < x_2$ and then decreases from one at $r = x_2$ to zero as $r \to \infty$. We now have a physical explanation for the range of integration in Eqn. (84).

5.3 Application to the Telecom process

Let $g(s) = (s)_+^{-\gamma}$, for $1 < \alpha < \gamma < 2$. Then $g(s)$ is strictly monotone decreasing on $(0, \infty)$, with $g'(s) = -\gamma\, s^{-\gamma-1}$ there. One has $A^\alpha = B^\alpha$ and thus the representations of Eqns. (80, 81) are valid and agree, but not the representations of Eqns. (83, 84) because $\int_0^\infty g(s)\, ds = \infty$ (an indication that the Telecom process is not differentiable). One can approximate the process by taking $\epsilon > 0$ and setting

$$g_\epsilon(s) = s^{-\gamma} \wedge \epsilon^{-\gamma} 1_{\{s \ge 0\}}$$

Now $\int_0^\infty g_\epsilon(s)\, ds = \epsilon^{1-\gamma}\gamma/(\gamma - 1) < \infty$ since $\gamma > 1$, so Eqns. (83, 84) with g replaced by g_ϵ determine a differentiable process $X_{t,\epsilon}^{(3)}$ that converges in probability to $X_t^{(2)}$ as $\epsilon \to 0$. Hence the following result holds for the Telecom process too:

Corollary 5.2 *The Telecom process Z_t of Eqn. (76) has the following equivalent representations:*

$$Z_t = \int_{\mathbb{R}} \int_{\mathbb{R}_+} \left[\int_0^t 1_{\{0 < x_1 < (r - x_2)_+^{-\gamma}\}} dr \right] M_\alpha(dx_1\, dx_2) \tag{87}$$

$$\stackrel{d}{=} \lim_{\epsilon \to 0} \int_0^t \left[\int_{\mathbb{R}} \int_0^{g_\epsilon(r - x_2)} M_\alpha(dx_1\, dx_2) \right] dr \tag{88}$$

$$\stackrel{d}{=} \lim_{\epsilon \to 0} \int_0^t \left[\int_{\mathbb{R}} \int_0^{g_\epsilon(r - x_2)} \int_{\mathbb{R}} u \tilde{N}(du\, dx_1\, dx_2) \right] dr \tag{89}$$

where $g_\epsilon(s) \equiv s^{-\gamma} \wedge \epsilon^{-\gamma} 1_{\{s \ge 0\}}$ and where M_α is an $S\alpha S$ random measure with control measure $dx_1\, dx_2$ and where $\tilde{N}$ is a compensated Poisson random measure with mean measure $|u|^{-\alpha-1}\, du\, dx_1\, dx_2$.

5.4 Example: fractional Brownian Motion

The Gaussian fBM has many different representations. We have already encountered two of them in Eqns. (18, 71). Setting $\alpha = 2$ and replacing $M_\alpha(dx_1\,dx_2)$ in Eqn. (88), by a Gaussian random measure $W(dx_1\,dx_2)$ (with $\mathbb{E}W(dx_1\,dx_2)^2 = dx_1\,dx_2$) and $-\gamma = 2H-3$, $H \in (\frac{1}{2}, 1)$, yields another equivalent representation for fBM. Thus,

$$B_H(t) \stackrel{d}{=} \lim_{\epsilon\to 0} \int_0^t \left[\int_{\mathbb{R}} \int_0^{g_\epsilon(r-x_2)} W(dx_1\,dx_2) \right] dr. \tag{90}$$

where $g_\epsilon(s) \equiv s^{2H-3} \wedge \epsilon^{2H-3} 1_{\{s\geq 0\}}$. The networking interpretation for this Gaussian-driven process is similar to that of the Poisson-driven process presented in Section 5.2: now $W(dx_1\,dx_2)$ represents a departure from the mean job rate initiated at time $x_2 \in \mathbb{R}$ and of duration τ satisfying $g_\epsilon(\tau) = x_1$, a random variable with Pareto distribution given by $\mathbb{P}[\tau > t] = g_\epsilon(t)/g_\epsilon(0) = (t/\epsilon)^{-\gamma}$ for $t \geq \epsilon$. Thus Eqn. (90) provides a networking explanation for fBM and helps explain why this process is so ubiquitous in telecommunications.

6 Conclusion

We have constructed families of stationary infinitely-divisible processes X'_t, of stationary-increment processes X_t, and of stationary random fields $X[\phi]$, useful for modeling network and telecommunications elements. These include and generalize a number of familiar processes already arising in telecommunications modeling including fractional Brownian motion and the Telecom process. Briefly, these models evolves as follows:

A Poisson random measure $N(du\,dx_1\,dx_2)$ generates random points (u, x_1, x_2) in $\mathbb{R}\times\mathcal{X}$. A point (u, x_1, x_2) indicates the arrival at time x_2 of a job request of magnitude $|u|$ and duration $\tau > 0$ satisfying $g(\tau) = x_1$. For any $\epsilon > 0$ and $\eta > 0$, the arrival times x_2 of "big" job requests of magnitude $|u| > \epsilon$ and "long" duration $\tau > \eta$ constitute a Poisson process with finite rate (so only finitely many of these arise in any compact time interval) proportional to $\nu\big([-\epsilon,\epsilon]^c\big)\, g(\eta)$ (proportional to $\epsilon^{-\alpha}\eta^{-\gamma}$ for the Telecom process).

The construction may be generalized easily to non-stationary processes, where these rates may vary in time, or to the case where job magnitudes and durations are dependent— one need only employ non-constant drift functions $m(t)$ and variance and Lévy measures $v(dx)$ and $\nu(du\,dx_1\,dx_2)$ that are not necessarily of product form $v(dx) = v\,dx$ and $\mu(du)\,dx_1\,dx_2$.

A Appendix: Poisson integrals

Let $N(dx)$ be a Poisson random measure on a space $\mathcal{X}$, with mean (or "control") measure $\nu(dx) = \mathbb{E}N(dx)$, and let $\tilde{N}(dx) \equiv N(dx) - \nu(dx)$ be the corresponding compensated Poisson measure. We collect here for convenience the conditions on $\nu(dx)$ and on an integrand function $f: \mathcal{X}\to\mathbb{R}$ to ensure that the following integrals would be well-defined, have well-defined means, or have finite variances:

$$
\begin{aligned}
N_1[f] &\equiv \int_{|f|\le 1} f(x)\,\tilde N(dx) + \int_{|f|>1} f(x)\,N(dx)\\
N[f] &\equiv \int_{\mathcal X} f(x)\,N(dx)\\
\tilde N[f] &\equiv \int_{\mathcal X} f(x)\,\tilde N(dx).
\end{aligned}
$$

Partially compensated Poisson integrals:

The *partially compensated* Poisson integral $N_1[f]$ is well-defined whenever

$$\int_{\mathcal X} \big(f(x)^2 \wedge 1\big)\nu(dx) < \infty.$$

If $\int_{\mathcal X} \big(f(x)^2 \wedge |f(x)|\big)\nu(dx) < \infty$, then $N_1[f]$ has well-defined mean $\mathbb{E}N_1[f] = \int_{|f|>1} f(x)\nu(dx)$. If $\int_{\mathcal X} f(x)^2\nu(dx) < \infty$, then $N_1[f]$ has finite variance $\mathrm{Var}N_1[f] = \int_{\mathcal X} f(x)^2\nu(dx)$. One can construct S$\alpha$S random variables of this form by taking $\mathcal X = \mathbb{R}$, $f(x) = x$, and $\nu(dx) \propto |x|^{-\alpha-1}$ for any $0 < \alpha < 2$; they have well-defined means (zero) only for $1 < \alpha < 2$, and none has a finite variance.

Uncompensated Poisson integrals:

The *uncompensated* Poisson integral $N[f]$ is well-defined whenever

$$\int_{\mathcal X} \big(f(x) \wedge 1\big)\nu(dx) < \infty.$$

If $\int_{\mathcal X} |f(x)|\nu(dx) < \infty$, then $N[f]$ has well-defined mean $\mathbb{E}N[f] = \int_{\mathcal X} f(x)\nu(dx)$. If $\int_{\mathcal X} \big(f(x)^2 \vee |f(x)|\big)\nu(dx) < \infty$, *i.e.*, if $f \in L^1 \cap L^2(\mathcal X, d\nu)$, then $N[f]$ has finite variance $\mathrm{Var}N[f] = \int_{\mathcal X} f(x)^2\nu(dx)$. One can construct S$\alpha$S random variables of this form by taking $\mathcal X = \mathbb{R}$, $f(x) = x$, and $\nu(dx) \propto |x|^{-\alpha-1}$ only for $0 < \alpha < 1$; none has a well-defined mean or finite variance.

Fully compensated Poisson integrals:

The *fully compensated* Poisson integral $\tilde N[f]$ is well-defined whenever

$$\int_{\mathcal X} \big(f(x)^2 \wedge |f(x)|\big)\nu(dx) < \infty.$$

It has mean $\mathbb{E}\tilde N[f] = 0$. If $\int_{\mathcal X} f(x)^2\nu(dx) < \infty$, then $\tilde N[f]$ has finite variance $\mathrm{Var}N[f] = \int_{\mathcal X} f(x)^2\nu(dx)$. One can construct S$\alpha$S random variables of this form by taking $\mathcal X = \mathbb{R}$, $f(x) = x$, and $\nu(dx) \propto |x|^{-\alpha-1}$ only for $1 < \alpha < 2$; they have well-defined means (namely zero) but infinite variance.

B Appendix: The Orlicz Space Φ_1

In this Appendix we verify that the OUL and fOUL processes of Section 3 are well-defined under the condition Eqn. (26). Denote by $\Phi_1 = \{f : \Psi(f) < \infty\}$ the collection of real-valued functions

$f : \mathcal{X} \to \mathbb{R}$ on some measure space $(\mathcal{X},\, dx)$ for which

$$\Psi(f) \equiv \int_{\mathcal{X}} \int_{\mathbb{R}} \big(|u\, f(x)| \wedge 1\big)\, \nu(du)\, dx$$

is finite, for some Lévy measure ν that satisfies the conditions

$$A(\nu) \equiv \int_{\mathbb{R}} \big(|u| \wedge 1\big)\, \nu(du) < \infty \qquad B(\nu) \equiv \int_{\mathbb{R}} \log_+ |u|\, \nu(du) < \infty. \tag{91}$$

Then we have

Proposition B.1 *Φ_1 is a linear space.*

Proof. For $f, g \in \Phi_1$, $u \in \mathbb{R}$,

$$\begin{aligned} |u(f+g)| \wedge 1 &\leq \big(|uf| + |ug|\big) \wedge 1 \\ &\leq \big(|uf| \wedge 1\big) + \big(|ug| \wedge 1\big) \end{aligned}$$

and, for $c \in \mathbb{R}$,

$$|u(c\, f)| \wedge 1 \leq \big(|c| \vee 1\big)\big(|u\, f| \wedge 1\big).$$

Upon integrating with respect to $\nu(du)\, dx$, we have

$$\Psi(c\, f + g) \leq (1 \vee |c|)\ \Psi(f) + \Psi(g) < \infty, \quad \text{so } c\, f + g \in \Phi_1.$$

∎

Note too that Ψ is monotonic in the sense that $\Psi(f) \leq \Psi(g)$ if $|f| \leq |g|$, so the conditions $g \in \Phi_1$ and $|f| \leq |g|$ for functions $f, g : \mathcal{X} \to \mathbb{R}$ imply $f \in \Phi_1$. In order to show that the fOUL of Section 3 is well defined under Eqn. (26), we first set $\mathcal{X} = \mathbb{R}$ (under Lebesgue measure dx) and show that $g, h \in \Phi_1$ for two specific functions g, h.

Proposition B.2 *If ν satisfies Eqn. (91) and $\mathcal{X} = \mathbb{R}$, then for any $\lambda > 0$ we have*

$$g(x) \equiv e^{-\lambda x} 1_{\mathbb{R}_+}(x) \in \Phi_1.$$

Proof.

$$\begin{aligned} \Psi(g) &\equiv \int_{\mathbb{R}_+} \int_{\mathbb{R}} \big(|u\, e^{-\lambda x}| \wedge 1\big)\, \nu(du) dx \\ &= \int_{\mathbb{R}} \int_0^{\lambda^{-1} \log_+ |u|} 1\, dx\, \nu(du) + \int_{\mathbb{R}} \int_{\lambda^{-1} \log_+ |u|}^{\infty} |u\, e^{-\lambda x}| dx\, \nu(du) \\ &= \lambda^{-1} \int_{\mathbb{R}} \log_+ |u|\, \nu(du) + \lambda^{-1} \int_{\mathbb{R}} |u|\, e^{-\log_+ |u|} \nu(du) \\ &= \lambda^{-1} \int_{\mathbb{R}} \big((\log_+ |u| + 1) \wedge |u|\big)\, \nu(du) \\ &= \lambda^{-1} [A(\nu) + B(\nu)] < \infty, \quad \text{so } g \in \Phi_1. \end{aligned}$$

∎

This shows that the OUL process of Section 3 is well-defined; now we turn to the fractional fOUL.

Proposition B.3 *If ν satisfies Eqn. (91) and $\kappa > 0$, then $h(x) \equiv x^{\kappa-1} 1_{(0,1)}(x) \in \Phi_1$.*

Proof.
If $\kappa \geq 1$ then $|h| \leq 1$ and hence $\Psi(h) \leq \Psi(1) = A(\nu) < \infty$ by Eqn. (91), so $h \in \Phi_1$. If $0 < \kappa < 1$, then

$$\begin{aligned}
|u| \geq 1 \Rightarrow \int_0^1 \big(|u\, x^{\kappa-1}| \wedge 1\big)\, dx &= 1 \\
|u| < 1 \Rightarrow \int_0^1 \big(|u\, x^{\kappa-1}| \wedge 1\big)\, dx &= \int_0^{|u|^{1/(1-\kappa)}} 1\, dx + \int_{|u|^{1/(1-\kappa)}}^1 |u| x^{\kappa-1}\, dx \\
&= |u|^{1/(1-\kappa)} + \kappa^{-1}\big[|u| - |u|^{1/(1-\kappa)}\big] \\
&\leq \kappa^{-1}\, |u|
\end{aligned}$$

so in either case the right-hand-side is bounded by $\kappa^{-1}\big(|u| \wedge 1\big)$ and

$$\begin{aligned}
\Psi(h) &\equiv \int_0^1 \int_{\mathbb{R}} \big(|u\, x^{\kappa-1}| \wedge 1\big)\, \nu(du) dx \\
&\leq \kappa^{-1} \int_{\mathbb{R}} \big(|u| \wedge 1\big)\, \nu(du) \\
&= \kappa^{-1}\, A(\nu) < \infty, \quad \text{so } h \in \Phi_1.
\end{aligned}$$

∎

Finally, we have

Proposition B.4 *If ν satisfies Eqn. (91), then $f \in \Phi_1$ for any function of the form*

$$f(x) = c\, x^{\kappa-1}\, e^{-\lambda x} 1_{\mathbb{R}_+}(x)$$

with $c > 0$, $\lambda > 0$, $\kappa > 0$, i.e., any such f satisfies

$$\int_{\mathcal{X}} \int_{\mathbb{R}} \big(|u\, f(x)| \wedge 1\big)\, \nu(du)\, dx < \infty.$$

Proof. For any function of the indicated form there are real numbers $a \geq 0$, $b \geq 0$ for which

$$\begin{aligned}
f(x) &\leq a e^{-\lambda x} 1_{\mathbb{R}_+}(x) + b x^{\kappa-1} 1_{(0,1)}(x) \\
&= a\, g(x) + b\, h(x)
\end{aligned}$$

for all $x \in \mathbb{R}$. Since $a\, g(x) + b\, h(x) \in \Phi_1$ by Propositions B.1–B.3, and $|f| \leq |a\, g(x) + b\, h(x)|$, we conclude $f \in \Phi_1$ by monotonicity. ∎

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