The Gamma Function

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COMPUTATION

In many of the probability distributions that arise in environmental modelling and data analysis—binomial, poisson, beta,chi-squared, exponential, gamma—the factorial function

$$n! \equiv n \times (n-1) \times \cdots \times 2 \times 1$$

or its continuous generalization the gamma function

$$\Gamma(z) \equiv \int_0^\infty t^{z-1} e^{-t}\, dt = (z-1)! \tag{1}$$

appears, and so in likelihood-based inference and estimation (including all Bayesian and most frequentist procedures) there may arise the need to evaluate $\Gamma(z)$ or its logarithm or their derivatives.

Euler's integral (1) (so-called because it arises from a formula Euler included in a 1729 letter to Goldbach) converges for any real $z > 0$, or indeed for any complex z with positive real part $\Re(z) > 0$, and by analytic continuation determines an analytic function of z away from the poles at the nonpositive integers that extends factorials from the positive integers to all complex numbers except the negative integers. Changing variables to $s = e^{-x}$ or $t = (2x)^{1/2}$ leads to equivalent integral expressions

$$\Gamma(z) = \int_0^1 \left(\ln \frac{1}{s}\right)^{z-1} ds = 2^{1-z} \int_0^\infty t^{2z-1} e^{-t^2/2}\, dt.$$

Artin [2] showed that $\Gamma(z)$ is the only smooth log-convex extension of the factorial function, so these integrals must coincide with the functions defined by Euler's limit

$$\Gamma(z) = \lim_{n\to\infty} \frac{1 \cdot 2 \cdot 3 \cdots n}{z(z+1)(z+2)\cdots(z+n)}\, n^z \tag{2}$$

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and by Weierstrass' infinite product

$$1/\Gamma(z) = ze^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}, \tag{3}$$

where $\gamma \equiv \lim_{m\to\infty}[1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m} - \ln m] \approx 0.577215664$ is the Euler-Mascheroni constant. From this it is easy to verify the Reflection Identity

$$z!(-z!) = \Gamma(1+z)\Gamma(1-z) = \frac{\pi z}{\sin \pi z}. \tag{4}$$

The Gamma Function grows astonishingly fast: $\Gamma(z)$ exceeds 10^{10}, 10^{100}, and 10^{1000} at z=15, 72, and 451, respectively. Because of this rapid growth the function is best evaluated as the exponential of its logarithm, $\Gamma(z) = \exp(\ln \Gamma(z))$. For integer or half-integer arguments the Gamma Function or its logarithm may be evaluated from the recursive relation $z\Gamma(z) = \Gamma(z+1)$ starting with $\Gamma(1) = 1$ or $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, while for noninteger arguments it may be evaluated using Binet's Integral Expression [11, §12.32]

$$\ln \Gamma(z) = \frac{\ln 2\pi}{2} + (z - \frac{1}{2}) \ln z - z + 2\int_0^\infty \frac{\arctan\left(\frac{t}{a}\right)}{e^{2\pi t} - z}\, dt$$

or the related Binet Series

$$\begin{aligned} \ln \Gamma(z) &= \frac{\ln 2\pi}{2} + (z - \frac{1}{2}) \ln z - z \qquad (5) \\ &+ \frac{1}{2}\left\{\frac{c_1}{(z+1)} + \frac{c_2}{2(z+1)(z+2)} + \frac{c_3}{3(z+1)(z+2)(z+3)} + \cdots\right\}, \end{aligned}$$

where $c_n = \int_0^1 (u+1)(u+2)\cdots(u+n-1)(2u-1)u\,du$ ($c_1 = \frac{1}{6}$, $c_2 = \frac{1}{3}$, $c_3 = \frac{59}{60}$, etc.) or by Stirling's Series [11, §12.33] by

$$\ln \Gamma(z) = \frac{\ln 2\pi}{2} + (z - \frac{1}{2}) \ln z - z - \sum_{i=1}^{\infty} \frac{(-1)^i B_i}{2i(2i-1)z^{2i-1}}, \tag{6}$$

where B_i is the i^{th} Bernoulli number [1, *p.* 804; $B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$, $B_3 = \frac{1}{42}$, $B_4 = \frac{1}{30}$, $B_5 = \frac{5}{66}$, etc.].

Each of these has simple error bounds (the error in (5) is bounded by $\frac{z+1}{2z}\left(\frac{z+1}{z+n+1}\right)^z$, while (6) is an alternating series containing $\ln \Gamma(z)$ between any two successive partial sums). The standard math library for System V-compliant Unix systems includes a numerical implementation `lgamma()` of the log Gamma function; other implementations in a variety of computer languages may be located with any World Wide Web search engine. For example, the TOMS Algorithm 542 [6, 7] (based on a rational approximation of Cody and Hillstrom [5]) is widely available on-line.

Frequently Stirling's Formula [1, *p.* 257]

$$\Gamma(z) \sim e^{-z} z^{z-1/2} (2\pi)^{1/2} \left[1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51840z^3} - \frac{571}{2488320z^4} + \ldots\right],$$

even to order zero or one in z^{-1}, gives adequate accuracy for applications.

DERIVATIVES

The logarithmic derivative $\psi(z) = \Gamma'(z)/\Gamma(z)$, called the *psi* or *digamma* function, arises in environmetrics too. Differentiating the logarithm of (3) gives

$$\psi(z+1) = \psi(z) + z^{-1} = -\gamma + \sum_{n=1}^{\infty} \frac{z}{n(n+z)}, \tag{7}$$

and in particular shows that $\psi(z)$ satisfies the indicated recurrence relation with initial value $\psi(1) = -\gamma$, while a second derivative gives

$$\psi'(z+1) = \psi'(z) - z^{-2} = \sum_{n=1}^{\infty} \frac{1}{n(n+z)^2},$$

the *trigama* function, with initial value $\psi'(1) = \pi^2/6$ and recurrence relation $\psi'(z+1) = \psi'(z) - z^{-2}$. Efficient algorithms for computing the digamma and trigamma functions are readily available [3, 9, 10].

EXAMPLE

Both log-normal and gamma distributions are commonly used to model continuous positive quantities like concentrations, masses, lifetimes, etc. *Sums* of independent gamma random variables (with the same rate parameter) again have gamma distributions, while *products* of independent log-normal random variables again have log-noraml distributions, so the gamma distribution is often preferred for quantities that may be viewed as sums or averages.

For example, in a model expressing the average DDT concentrations (measured in parts-per-million, or PPM) in N fish $\{Y_i\}_{i\in I}$ as independent Gamma random variables with common shape parameter α and rate (inverse scale) λ, i.e., with density $f_i(y)$ given by

$$Y_i \sim \mathsf{Ga}(\alpha, \lambda): \qquad f_i(y) = \lambda^\alpha y^{\alpha-1} e^{-\lambda y}/\Gamma(\alpha),\ \ y > 0,$$

the log likelihood function for the two parameters is

$$\ell(\alpha, \lambda) = N\alpha \ln \lambda + (\alpha - 1)\sum_{i\in I} \ln Y_i - \lambda \sum_{i\in I} Y_i - N \ln \Gamma(\alpha)$$

with partial derivatives

$$\frac{\partial \ell}{\partial \alpha} = N\big[\ln \lambda - \psi(\alpha)\big] + \sum_{i\in I} \ln Y_i \qquad\qquad \frac{\partial \ell}{\partial \lambda} = \frac{N\alpha}{\lambda} - \sum_{i\in I} Y_i$$

and Maximum Likelihood Estimates (MLE's) satisfying $\hat{\lambda} = \hat{\alpha}/\bar{Y}$ and

$$\psi(\hat{\alpha}) - \ln(\hat{\alpha}) = \frac{1}{N}\sum_{i\in I} \ln Y_i - \ln\left(\frac{1}{N}\sum Y_i\right) = \overline{\ln Y} - \ln \overline{Y},$$

unavailable in closed form but easily obtained from computed values of $\psi(z)$.

The Information Matrix (both expected and observed) for the Gamma distribution is

$$I(\alpha, \lambda) = \begin{bmatrix} \psi'(\alpha) & -\lambda^{-1} \\ -\lambda^{-1} & \alpha\lambda^{-2} \end{bmatrix}$$

with determinant $\Delta(\alpha, \lambda) = \lambda^{-2}[\alpha\psi'(\alpha) - 1]$, so the MLE's for α and λ will have asymptotic variances $\alpha/N[\alpha\psi'(\alpha) - 1]$ and $\lambda^2\psi'(\alpha)/N[\alpha\psi'(\alpha) - 1]$, respectively, with asymptotic correlation coefficient $[\alpha\psi'(\alpha)]^{-1/2}$, while a noninformative Bayesian analysis [4, *p.* 54] might begin with the Jeffreys' prior distribution $\pi_J(\alpha, \lambda) \propto \lambda^{-1}[\alpha\psi'(\alpha) - 1]^{1/2}$ [8]. In a similar fashion estimates and inference for the Beta, Dirichlet, Chi-Squared, and other distributions all depend on evaluation of the Gamma function and its logarithm and their first two derivatives.

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